Topics in finite graph Ramsey theory

by

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Abstract

For a positive integer r and graphs F, G, and H, the graph Ramsey arrow notation $F \longrightarrow (G)^H_r$ means that for every r-colouring of the subgraphs of F isomorphic to H, there exists a subgraph G' of F isomorphic to G such that all the subgraphs of G' isomorphic to H are coloured the same. Graph Ramsey theory is the study of the graph Ramsey arrow and related arrow notations for other kinds of "graphs" (*e.g.*, ordered graphs, or hypergraphs). This thesis surveys finite graph Ramsey theory, that is, when all structures are finite.

One aspect surveyed here is determining for which G, H, and r, there exists an F such that $F \longrightarrow (G)^H_r$. The existence of such an F is guaranteed when H is complete, whether "subgraph" means weak or induced, and existence results are also surveyed when H is non-complete. When such an F exists, other aspects are surveyed, such as determining the order of the smallest such F, finding such an F in some restricted family of graphs, and describing the set of minimal such F's.

Acknowledgements

I would like to first thank God for helping me get through this, and for putting all the following people in my life to make this possible. To my constant support, my loving and unbelievably patient wife, I give my undying appreciation and affection. I couldn't have done it without you. A heartfelt thanks to my supervisor Dr. David Gunderson for his dedication to helping me, his constant example of hard work, his insistence of the highest quality, and his financial support. I also am very grateful to Dr. Michael Doob, Dr. William Kocay, and Dr. Craig Platt for their time and effort devoted to this thesis. I would also like to thank my mother for her encouragement and support, Karen Johannson for her frequent assistance, and Michael Potter for his advice and his time.

Dedication

I would like to dedicate this thesis to my grandparents Leif and Victoria Borgersen, who have always believed in me.

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Chapter 1

Introduction

This introduction briefly introduces Ramsey theory on graphs and outlines the structure of this thesis.

1.1 Notation

Unless otherwise stated, all variables are integers. Notation used in this thesis is, for the most part, standard. For notation used in graph theory, see Appendix A. Denote the integers by $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$, and the positive integers by $\mathbb{Z}^+$. For any $a \leq b$, let $[a, b] = \{x \in \mathbb{Z} : a \leq x \leq b\}$. The cardinality of a set S is denoted $|S|$, a set containing k elements is called a k-set, and the set of all k-sets contained in some set S is denoted $[S]^k = \{S' \subseteq S : |S'| = k\}$. For simplicity, write $[a,b]^k$ for $[[a,b]]^k$.

For any set S, and any $r \in \mathbb{Z}^+$, an *r-partition*of S is a set $\{S_1, \ldots, S_r\}$ of disjoint subsets of S such that $S_1 \cup \cdots \cup S_r = S$. The S_i's are called *partite sets* (or *partition classes*) and (in this thesis) are usually non-empty. If S and C are sets with $|C| = r \geq 2$, any function $\Delta : S \to C$ is an *r-colouring* of S, and the elements of C are called *colours*. In Ramsey theory, the Greek letter Δ is often used to denote a colouring. For each $i \in C$, $\Delta^{-1}(i) \subseteq S$ is called a *colour class*, and any subset of a colour class is said to be *monochromatic*. For example, if $S = \{a, b, c\}$, $C = \{\text{red}, \text{blue}\}$, and $\Delta(a) = \Delta(c) = \text{red}$, and $\Delta(b) = \text{blue}$, then the colour classes are $\{a, c\}$ and $\{b\}$. Note that every r-colouring induces an r-partition, though many distinct r-colourings induce the same r-partition. Often the elements of C are not important, and so for convenience, when $r = 2$, $C = \{\text{red}, \text{blue}\}$ is commonly used, and when $r > 2$, $C = [1, r]$ is commonly used. The letter r is used throughout this thesis to denote a number of colours, and in such a context should always be at least two to avoid trivialities.

1.2 Pigeonhole principle and Ramsey's theorem

The pigeonhole principle is a basic tool in Ramsey theory, and is itself now considered the simplest "Ramsey-type" theorem.

Theorem 1.2.1 (Pigeonhole principle)**.** *If at least $mr + 1$ objects are partitioned into r (possibly empty) subsets, at least one subset contains $m + 1$ elements.*

The pigeonhole principle can be restated in various ways, *e.g.*:

(a) If S is a set with $|S| \geq mr + 1$, and $S = S_1 \cup \cdots \cup S_r$ is a partition of S, then there exists $i \in [1, r]$ such that $|S_i| \geq m + 1$.

(b) If $\Delta : [1, mr+1] \to [1, r]$, then there exists $i \in [1, r]$ such that $|\Delta^{-1}(i)| \geq m+1$.

Theorem 1.2.2 (Infinite pigeonhole principle)**.** *For every finite colouring of an infinite set, there exists a colour class which is infinite.*

Two theorems due to Ramsey generalize the finite and infinite versions of the pigeonhole principle to the colouring of k-sets, rather than just singletons (the proofs appear in Chapter 2).

Theorem 1.2.3 (Ramsey's theorem for finite sets, 1930 [126])**.** *For all $m, k, r \in \mathbb{Z}^+$, there exists $n \in \mathbb{Z}^+$ such that for every n-set N, and for every r-colouring $\Delta : [N]^k \to [1, r]$ of the k-subsets of N, there exists $M \in [N]^m$ such that $[M]^k$ is monochromatic.*

Denote the least such n by $R_k(m;r)$. Note that since $\mathbb{Z}^+$ is well-ordered, the existence of such an n is equivalent to the existence of a least such n. Therefore, from this point on, theorems which guarantee the existence of a positive integer are stated as guaranteeing the existence of the least such positive integer.

Theorem 1.2.4 (Ramsey's theorem for infinite sets, 1930 [126]). *For every $k, r \in \mathbb{Z}^+$, every infinite set X, and every $\Delta : [X]^k \to [1, r]$, there exists an infinite set $Y \subseteq X$ such that $[Y]^k$ is monochromatic.*

When $k = 1$, the two versions of Ramsey's theorem are exactly the two versions of the pigeonhole principle. Ramsey's theorem for finite sets, referred to from this point on as simply "Ramsey's theorem", is central to this thesis. In Ramsey theory, it is common practice to put a semicolon before the number of colours (*e.g.*, $R_k(m;r)$). While semicolons may be used in different contexts as well, the number of colours will always be separated by a semicolon.

An "arrow notation" (known as a "Ramsey arrow") is used to simplify statements that are similar to Ramsey's theorem. This Ramsey arrow was introduced by Erdős and Rado in 1953 [46]. For positive integers n, m, k and r, write

$$n \longrightarrow (m)^k_r \tag{1}$$

if for every n-set N, and for every $\Delta : [N]^k \to [1, r]$, there exists $M \in [N]^m$ such that $[M]^k$ is monochromatic. At this point it can be pointed out that Ramsey-type theorems can be hard to read and understand at first due to the number of alternating quantifiers. For example, there are a total of four quantification switches in Ramsey's theorem (stated below for colouring the integers):

$$\forall m, k, r \in \mathbb{Z}^+, \exists n \in \mathbb{Z}^+ \text{ s.t. } \forall \Delta : [1,n]^k \to [1,r], \exists i \in [1,r] \text{ and } S \in [1,n]^m \text{ s.t.}$$
$$\forall S' \in [S]^k, \Delta(S') = i.$$

Using the arrow notation, Ramsey's theorem can be restated briefly:

Theorem (Ramsey's theorem restated)**.** *For all $m, k, r \in \mathbb{Z}^+$, there exists a least integer $n = R_k(m; r)$ such that $n \longrightarrow (m)^k_r$.*

1.3 Ramsey's theorem for graphs

The arrow notation (1) has been generalized to graphs in many ways. The main idea is to let F be a "large" graph, G be a "medium sized" graph, and H a "small" graph (see Figure 1.1). For now, a "copy" of H in G means a subgraph (either weak or induced depending on context) of G isomorphic to H. If for every r-colouring of the copies of H in F, there is a copy G' of G in F such that every copy of H in G' is the same colour, then write

$$F \longrightarrow (G)^H_r. \tag{2}$$

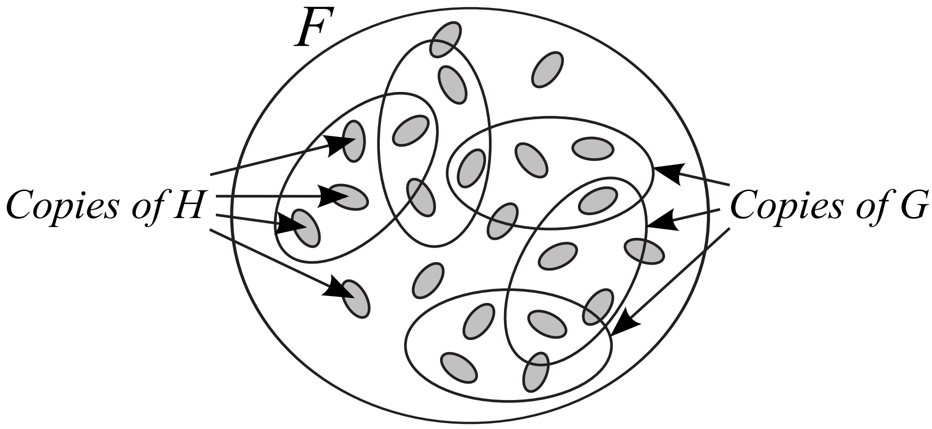


Figure 1.1: Visualizing Ramsey theory for graphs

The field of "Graph Ramsey theory" is essentially the study of aspects of various graph Ramsey arrows. For any $n \in \mathbb{Z}^+$, let K_n denote the complete graph on n vertices (every pair of vertices is an edge). Since the copies of K_k inside K_n are in one-to-one correspondence with the k-subsets of an n-set, Ramsey's theorem can be restated in terms of graphs.

Theorem (Ramsey's theorem—graph theoretic version)**.** *For all $m, k, r \in \mathbb{Z}^+$, there exists a least integer $n = R_k(m; r)$ such that*

$$K_n \longrightarrow (K_m)_r^{K_k}.$$

There are two camps of thought regarding this arrow notation (2): those who primarily work in the induced case, and those who primarily work in the weak (or not necessarily induced) case. Neither group wants to use extra notation, and so those of the induced camp use "$\longrightarrow$" for the induced case, while those studying in the weak area use "$\longrightarrow$" for the weak case. The standard is often viewed as using "$\longrightarrow$" for the induced case (see, *e.g.*, [102]). However, the weak case is discussed more in this work than the induced case, and therefore "$\longrightarrow$" is used in this thesis when all subgraphs are weak. Therefore, in this work, the arrow notation (2) is referred to as the "weak graph Ramsey arrow". When the subgraphs of interest must be induced, another arrow notation

$$F \xrightarrow{\text{ind}} (G)^H_r \tag{3}$$

is used, referred to as the "induced graph Ramsey arrow" (also known as the "strong graph Ramsey arrow").

1.4 Structure of this thesis

All results presented in this thesis are found in the literature.

Some proofs of Ramsey's theorem are presented first in Chapter 2. Chapter 3 surveys results on finding and bounding the numbers $R_k(m;r)$ (defined in Ramsey's theorem). As well, in Chapter 3, an "off-diagonal" version of the Ramsey numbers is introduced. For any $k, r, m_1, \ldots, m_r \in \mathbb{Z}^+$, let $R_k(m_1, \ldots, m_r)$ denote the least integer n such that for every n-set N, and for every $\Delta : [N]^k \to [1,r]$, there exists $i \in [1,r]$ and $M_i \in [N]^{m_i}$ such that $[M_i]^k$ is monochromatic under Δ. Chapter 3 also surveys known results on finding and bounding the numbers $R_k(m_1, m_2)$.

For any $r \in \mathbb{Z}^+$, and for any graphs G and H, let $R(G; H; r)$ denote the least integer n (if such an integer exists) such that

$$K_n \longrightarrow (G)_r^H$$

(an additional semicolon is used in the notation $R(G; H; r)$, but a semicolon is still used to separate the number of colours). Ramsey's theorem then says that for all $k, m, r \in \mathbb{Z}^+$, $R(K_m; K_k; r)$ exists (since $R(K_m; K_k; r) = R_k(m; r)$). Note that for any graph G on m vertices, $G \subseteq K_m$, and therefore if $n = R(K_m; K_k; r)$ then $K_n \longrightarrow (G)_r^{K_k}$. Thus for any graph G with m vertices, and any $k, r \in \mathbb{Z}^+$, $R(G; K_k; r)$ exists, and $R(G; K_k; r) \leq R(K_m; K_k; r)$. The numbers $R(G; H; r)$, when they exist, are known as *weak graph Ramsey numbers*. In Chapter 4, the problem of determining for any graph G the value $R(G; K_2; 2)$ is discussed, the numbers $R(G; K_k; r)$ are generalized to an off-diagonal version, and some results are surveyed for determining values in this off-diagonal version.

Note: throughout this thesis, in order to prove that a least positive integer X exists satisfying some Ramsey property (as is done above), a phrase similar to the following is written: "it suffices to show that $X \leq Y$". This means that it suffices to show that the positive integer Y satisfies the same Ramsey property, and since X is the least such positive integer, it follows that X exists and $X \leq Y$.

The main result of Chapter 5 is that for all $k, r \in \mathbb{Z}^+$, and for any graph G, there exists a graph F such that $F \xrightarrow{\text{ind}} (G)_r^{K_k}$. Little is known about the minimum number of vertices in such an F when G is not complete.

Arrow notations (both weak and strong) may also be generalized to hypergraphs, ordered graphs, and ordered hypergraphs in natural ways. Another variation of the arrow notation for graphs used in this thesis is adding the subscript "part" to an arrow notation (*e.g.*, "$\longrightarrow_{\text{part}}$"), which means that for some $k \in \mathbb{Z}^+$, only k-partite graphs and subgraphs are of interest (F, G, and H are all k-partite). Another arrow "$\xrightarrow{\text{ind}}_{\text{ord}}$" introduced in Section 5.8 deals with Ramsey theory questions regarding orderings rather than colourings (rather than colouring ordered hypergraphs as one might expect).

Some needed concepts of *extremal graph theory* are given in Chapter 6. Chapter 7 considers families of graphs (*e.g.*, graphs with bounded degree) whose weak graph Ramsey numbers grow linearly.

For a graph G, and $r \in \mathbb{Z}^+$, the problem considered in Chapter 8 is finding a graph F contained in some restricted set of graphs (*e.g.*, triangle-free graphs) such that $F \longrightarrow (G)_r^{K_2}$. Such questions form so-called "restricted graph Ramsey theory".

The problem of describing the set of (vertex or edge) minimal graphs F so that $F \longrightarrow (G)_r^{K_2}$ is discussed in Chapter 9.

Before Chapter 10, the subgraphs being coloured (the H's) are complete. When H is not complete, there are examples of G, H, and r such that for every graph F, $F \not\longrightarrow (G)^H_r$. Chapter 10 contains a theorem characterizing those triples (G, H, r) for which there exists a graph F such that $F \longrightarrow (G)^H_r$ (Theorem 10.2.3).

Chapter 2

Ramsey's theorem

2.1 Introduction

Ramsey theory is named for Frank Plumpton Ramsey (1903–1930), a British mathematician who attended the University of Cambridge. For accounts of Ramsey's life, and contributions to science, see *e.g.*, [10, 81, 91, 97, 98], and for collections of Ramsey's works, see [127, 128, 129]. For convenience, the finite version of Ramsey's theorem is repeated here.

Theorem (Ramsey's theorem, given as Theorem 1.2.3)**.** *For all $m, k, r \in \mathbb{Z}^+$, there exists a least integer $n = R_k(m; r)$ such that for every n-set N, and for every r-colouring $\Delta : [N]^k \to [1, r]$, there exists $M \in [N]^m$ such that $[M]^k$ is monochromatic.*

The purpose of this chapter is to present three proofs of Ramsey's theorem for finite sets (none of which seem to follow Ramsey's original proof of Ramsey's theorem for finite sets), a proof of Ramsey's theorem for infinite sets, a closely related problem in geometry (known as the "n-gon problem"), and the Paris-Harrington theorem, a generalization of Ramsey's theorem which demonstrates a surprising fact from logic that there are statements true in Peano arithmetic that are not provable within Peano arithmetic.

2.2 Direct proof by induction

The idea in the following first proof of Ramsey's theorem (for finite sets), follows the same idea as the original proof of Ramsey's theorem for infinite sets, and therefore, this proof is usually attributed to Ramsey himself, though he did not explicitly present it.

First proof of Ramsey's theorem (as given by Nešetřil [102, p. 1334]). This proof of Ramsey's theorem is by induction on k. For each $k \in \mathbb{Z}^+$, let $S(k)$ be the statement that for all $m, r \in \mathbb{Z}^+$, $R_k(m;r)$ exists (note that when $m = 1$ or $r = 1$, the theorem is trivial, so one can assume that $m \geq 2$ and $r \geq 2$).

Base Case: By the pigeonhole principle, $R_1(m;r) = r(m-1)+1$, and therefore

$S(1)$ holds.

Inductive Step: Let $k \geq 2$, and assume that $S(k-1)$ holds. Let $m, r \in \mathbb{Z}^+$ and let $t = R_1(m;r)$. For $i \in [1,t]$, define s_i recursively by setting $s_1 = 1$, and for each $i \in [1, t-1]$, let $s_{i+1} = R_{k-1}(s_i; r) + 1$. It suffices to show that $R_k(m;r) \leq s_t$.

Let $n = s_t$, let N be a well-ordered n-set, and let $\Delta : [N]^k \to [1,r]$. In this proof, a set $Y = \{y_1 < y_2 < \cdots < y_t\} \in [N]^t$ is constructed inductively so that there exists a set $Y' \in [Y]^m$ that is monochromatic under Δ. Let $y_1 = \min N$, and define $\Delta_1 : [N \setminus \{y_1\}]^{k-1} \to [1,r]$ by $\Delta_1(A) = \Delta(A \cup \{y_1\})$. By the definition of $n = R_{k-1}(s_{t-1}; r) + 1$, there exists a set $X_1 \subseteq N \setminus \{y_1\}$ such that $|X_1| = s_{t-1}$ and $[X_1]^{k-1}$ is monochromatic in the colour c_1 under Δ_1.

Let $i \in [1, t-1]$, and assume that $y_1, \ldots, y_i \in N$, $c_1, \ldots, c_i \in [1,r]$, and $X_i \subseteq \cdots \subseteq X_1 \subseteq N$ have already been defined such that $|X_i| = s_{t-i} = R_{k-1}(s_{t-i-1}; r) + 1$. Let $y_{i+1} = \min X_i$, and define $\Delta_{i+1} : [X_i \setminus \{y_{i+1}\}]^{k-1} \to [1,r]$ by $\Delta_{i+1}(X) = \Delta(X \cup \{y_{i+1}\})$. Then by the definition of s_{t-i}, there exists $X_{i+1} \subseteq X_i \setminus \{y_{i+1}\}$ such that $|X_{i+1}| = s_{t-i-1}$ and $[X_{i+1}]^{k-1}$ is monochromatic in the colour c_{i+1} under Δ_{i+1}. This completes the inductive definition of $Y = \{y_1, y_2, \ldots, y_t\}$ (see Figure 2.1).

Let $\Delta_Y : Y \to [1,r]$ be defined by $\Delta_Y(y_i) = c_i$. Then by the definition of t, there exists $Y' \subseteq Y$, $|Y'| = m$, such that Y' is monochromatic under Δ_Y. Then by the construction of the y_i's, $[Y']^k$ is monochromatic under Δ. $\square$

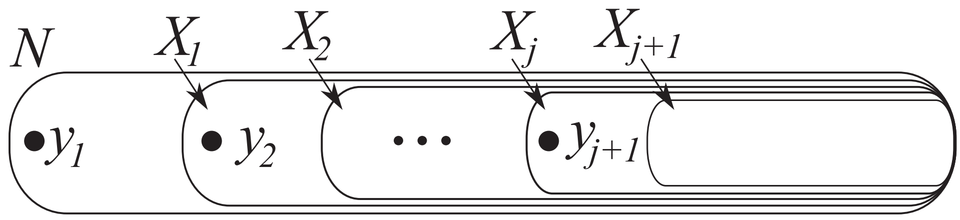


Figure 2.1: Constructing the set $\{y_1, y_2, \ldots\}$

2.3 Ramsey's theorem for infinite sets

Ramsey's theorem for infinite sets is restated for convenience, and is proved in this section. The proof given here nearly duplicates the first proof of Ramsey's theorem given in Section 2.2.

Theorem (Ramsey's theorem for infinite sets, given as Theorem 1.2.4)**.** *For every $k, r \in \mathbb{Z}^+$, every infinite set X, and every $\Delta : [X]^k \to [1, r]$, there exists an infinite set $Y \subseteq X$ such that $[Y]^k$ is monochromatic.*

Proof. This proof is by induction on k. For any $k \in \mathbb{Z}^+$, let $S(k)$ be the statement that for all $r \in \mathbb{Z}^+$, for any countably infinite set X, and for every r-colouring $\Delta : [X]^k \to [1, r]$, there exists an infinite set $Y \subseteq X$ such that Δ is constant on $[Y]^k$.

Base Case: The statement $S(1)$ is the infinite pigeonhole principle, and thus $S(1)$ holds.

Inductive Step: Let $k \geq 1$, and assume that $S(k)$ holds. It remains to show that

$S(k+1)$ holds. Let $X = \{x_1 \leq x_2 \leq x_3 \leq \cdots\}$ be a countable well-ordered infinite set. Let $\Delta : [X]^{k+1} \to [1, r]$. An infinite sequence of vertices $y_1, y_2, \ldots \in X$ is constructed inductively such that $Y = \{y_1, y_2, \ldots\}$ contains an infinite set Y' such that $[Y']^{k+1}$ is monochromatic under Δ.

Let $y_1 = x_1$. Define the colouring $\Delta_1 : [X \setminus \{y_1\}]^k \to [1, r]$ as follows: for $P \in [X \setminus \{y_1\}]^k$, $\Delta_1(P) = \Delta(P \cup \{y_1\})$. Then, since $S(k)$ is true, there exists an infinite set $X_1 \subseteq X \setminus \{y_1\}$ such that Δ_1 is constant on $[X_1]^k$, and therefore Δ is constant, in say colour c_1, on $\{A \cup \{y_1\} : A \in [X_1]^k\}$.

Let $j \geq 1$, and assume that y_j, X_j, c_j, and Δ_j have already been defined. Let y_{j+1} be the least element in X_j. Define the colouring $\Delta_{j+1} : [X_j \setminus \{y_{j+1}\}]^k \to [1, r]$ as follows: for $P \in [X_j \setminus \{y_{j+1}\}]^k$, let $\Delta_{j+1}(P) = \Delta(P \cup \{y_{j+1}\})$. Then since $S(k)$ is true, there exists an infinite set $X_{j+1} \subseteq X_j \setminus \{y_{j+1}\}$ such that Δ_{j+1} is constant on $[X_{j+1}]^k$, and therefore Δ is constant on $\{A \cup \{y_{j+1}\} : A \in [X_{j+1}]^k\}$. This completes the inductive construction.

Let $Y = \{y_1, y_2, \ldots\}$, and let $\Delta_Y : Y \to [1, r]$ be defined as $\Delta_Y(y_i) = c_i$. By the infinite pigeonhole principle, there exists an infinite set $Y' \subseteq Y$ monochromatic under Δ_Y. By the construction of the y_i's, Δ is constant on $[Y']^{k+1}$. Therefore $S(k+1)$ holds.

Therefore by mathematical induction, for all $n \in \mathbb{Z}^+$, $S(n)$ holds. $\square$

Ramsey's theorem for infinite sets can be reworded in terms of graph theory. Let $K_{\aleph_0}$ denote the complete graph with a countably infinite vertex set.

Theorem (Ramsey's theorem for infinite sets–restated for graphs)**.** *For all $k, r \in \mathbb{Z}^+$, $K_{\aleph_0} \longrightarrow (K_{\aleph_0})_r^{K_k}$.*

2.4 Two colours implies r colours

In order to simplify the next two proofs of Ramsey's theorem, the following proposition is presented first. The proof uses a so-called "colour grouping" argument. For convenience, let $R_k(m)$ denote $R_k(m; 2)$.

Proposition 2.4.1 (Ramsey, 1930 [126])**.** *Let $k \in \mathbb{Z}^+$. If for all $m \in \mathbb{Z}^+$, $R_k(m)$ exists, then for all $r > 2$, $R_k(m; r)$ exists as well.*

Proof. Let $k \in \mathbb{Z}^+$, and assume that for all $m \in \mathbb{Z}^+$, $R_k(m)$ exists. The proof of Proposition 2.4.1 is by induction on r. For any $r \geq 2$, let $S(r)$ be the statement that for all $m \in \mathbb{Z}^+$, $R_k(m; r)$ exists.

Base Case: The above assumption is exactly $S(2)$.

Inductive Step: Let $r \geq 2$, and assume that $S(r)$ holds. By $S(2)$, let $M = R_k(m)$, and by $S(r)$, let $n = R_k(M; r)$. To show that $R_k(m; r+1)$ exists, it suffices to show that for $n = R_k(M; r)$, that $n \longrightarrow (m)_{r+1}^k$. Let X be any n-set, and let

$\Delta : [X]^k \to [1, r+1]$. Define $\Delta' : [X]^k \to [1, r]$ by

$$\Delta'(T) = \begin{cases} \Delta(T) & \text{if } \Delta(T) \in [1, r-1], \\ \\ r & \text{if } \Delta(T) \in [r, r+1]. \end{cases}$$

By definition of n, there exists an $Y \in [X]^M$ and $i \in [1, r]$ such that $[Y]^k$ is monochromatic in the i-th colour. If $i \in [1, r-1]$, then $[Y]^k$ is monochromatic under Δ as well, which is more than sufficient to prove the proposition. If $i = r$, then Δ restricted to $[Y]^k$ is a 2-colouring, and by the definition of M, there exists a $Y' \in [Y]^m$ monochromatic under Δ. Therefore $S(r+1)$ holds.

Thus by induction, for all $r \in \mathbb{Z}^+$, $S(r)$ holds. □

Proposition 2.4.1 shows that, in order to prove Ramsey's theorem, it suffices to prove only the $r = 2$ case. The above colour graphing argument is commonly used in Ramsey theory and can be used in many other situations.

2.5 Infinite implies finite

A second proof of Ramsey's theorem that uses Ramsey's theorem for infinite sets, together with a standard theorem from graph theory known as König's infinity lemma, is presented here.

Recall that an infinite tree is *locally finite* if and only if the degree of every vertex is finite. For the definition of a rooted tree, and other standard definitions in graph theory, the reader is referred to Appendix A.

Theorem 2.5.1 (König's infinity lemma, 1927 [86]). *Every infinite locally finite rooted tree contains an infinite path starting from the root.*

Proof. Let T be an infinite locally finite tree, rooted at v_1. By the infinite pigeonhole principle, at least one branch of T (a component of $T \setminus \{v_1\}$), call it T_1, is infinite. Let v_2 be the vertex in $N(v_1) \cap V(T_1)$. Then T_1 is an infinite locally finite tree rooted at v_2.

Let $k \geq 1$, and assume that the path $v_1, \ldots, v_k$ has already been constructed, and the infinite locally finite tree T_{k-1} has been defined, rooted at v_k. By the infinite pigeonhole principle, at least one branch of T_{k-1}, call it T_k, is infinite. Let v_{k+1} be the vertex in $N(v_k) \cap V(T_k)$. This completes the inductive construction, and the set $\{v_1, v_2, \ldots\}$ forms an infinite path starting at the root. $\square$

The following proof seems to have been first mentioned by Erdős and Szekeres [48] in 1935.

Second proof of Ramsey's theorem. By Proposition 2.4.1, it suffices to prove the $r = 2$ case. Let $k, m \in \mathbb{Z}^+$. For any set X, and any 2-colouring $\Delta : [X]^k \to$

{red, blue}, Δ is a *bad colouring of* X iff there is no $M \in [X]^m$ monochromatic under Δ. By Ramsey's theorem for infinite sets, there are no bad colourings when X is an infinite set.

Assume that for some particular $k, m \in \mathbb{Z}^+$, $R_k(m)$ does not exist. Then for any finite set X, there exists a bad colouring of X (by definition, the "empty colouring", the colouring which has the empty set as its domain, is also considered a bad colouring). Consider the sets $\{[1,n] : n \in \mathbb{Z}^+\}$. For any bad colouring Δ on $[1,n]^k$, $\Delta' = \Delta|_{[1,n-1]^k}$ is a bad colouring on $[1,n-1]^k$. Let T be the tree on the vertex set

$$V(T) = \{\Delta : \Delta \text{ is a bad colouring on } [1,t],\, t \in \mathbb{Z}^+ \cup \{0\}\},$$

rooted at the empty colouring, with edge set defined as follows: if for some $i, j \in \mathbb{Z}^+$, $i \leq j$, Δ_1 is a bad colouring on $[1,i]$ and Δ_2 is a bad colouring on $[1,j]$, then $\{\Delta_1, \Delta_2\} \in E(T)$ if and only if $j = i+1$ and $\Delta_2|_{[1,i]} = \Delta_1$. Then T is an infinite locally finite rooted tree. Therefore by König's infinity lemma (Theorem 2.5.1), there exists an infinite path $\Delta_0, \Delta_1, \ldots$ starting at the root (the empty colouring). This infinite branch corresponds to a bad colouring $\Delta_\infty : \mathbb{Z}^+ \to \{\text{red}, \text{blue}\}$, which extends each of the bad colourings $\Delta_0, \Delta_1, \ldots$. The existence of Δ_∞ is a contradiction to Ramsey's theorem for infinite sets (Theorem 1.2.4). $\square$

2.6 The off-diagonal Ramsey numbers

The next proof of Ramsey's theorem presented in Section 2.7 involves the following generalizations of the Ramsey arrow for integers, and the Ramsey number $R_k(m)$. For $n, k, a, b \in \mathbb{Z}^+$, write $n \longrightarrow (a, b)^k_2$ if for any n-set X, and for any 2-colouring $\Delta : [X]^k \to \{\text{red}, \text{blue}\}$, there exists either $A \in [X]^a$ such that $[A]^k$ is monochromatic red, or $B \in [X]^b$ such that $[B]^k$ is monochromatic blue. Let $R_k(a, b)$ denote the least n (if it exists) such that $n \longrightarrow (a, b)^k_2$. When $a \neq b$, the numbers $R_k(a, b)$ are sometimes referred to as the "off-diagonal" Ramsey numbers (note that $R_k(m) = R_k(m, m)$). The following equivalent off-diagonal version of Ramsey's theorem is proved below (the $k = 2$ case on page 23, and the general case on page 24).

Theorem 2.6.1 (Ramsey's theorem–off-diagonal version)**.** *For all $k, a, b \in \mathbb{Z}^+$, $R_k(a, b)$ exists.*

Proving Theorem 2.6.1 proves Ramsey's theorem since for all $k, m \in \mathbb{Z}^+$, $R_k(m) = R_k(m, m)$, and Ramsey's theorem implies the off-diagonal version since for all $a, b \in \mathbb{Z}^+$, $R_k(a, b) \leq R_k(\max\{a, b\})$.

The off-diagonal version of Ramsey's theorem can be restated for graphs. For graphs G and H, let $\binom{G}{H}$ denote the set of weak (not necessarily induced) subgraphs of G that are isomorphic to H (see Appendix A for further graph theoretic notation). Note that in some works $\binom{G}{H}$ is used to denote the set of induced subgraphs of G

isomorphic to H (see *e.g.*, [102]).

For any graphs F, G_1, G_2 and H, write

$$F \longrightarrow (G_1, G_2)_2^H$$

if for every $\Delta : \binom{F}{H} \longrightarrow \{\text{red}, \text{blue}\}$, there exists either $G_1' \in \binom{F}{G_1}$ such that $\binom{G_1'}{H}$ is monochromatic red, or there exists $G_2' \in \binom{F}{G_2}$ such that $\binom{G_2'}{H}$ is monochromatic blue. One can say that "G is monochromatic" when what is meant is that $\binom{G}{H}$ is monochromatic.

Theorem (Theorem 2.6.1—restated for graphs)**.** *For all $k, a, b \in \mathbb{Z}^+$, there exists a least $n = R_k(a, b)$ such that $K_n \longrightarrow (K_a, K_b)_2^{K_k}$.*

2.7 The Erdős-Szekeres recursion

Using the off-diagonal generalization of the Ramsey numbers defined in Section 2.6, Erdős and Szekeres produced the following recursive bound, presented first for the $k = 2$ case (which corresponds to the colouring of edges), and then for the general k case. Abbreviate $R_2(a, b)$ by simply $R(a, b)$, the most studied off-diagonal Ramsey numbers.

2.7.1 Colouring edges

Theorem 2.7.1 (Erdős-Szekeres recursion ($k = 2$ version), 1935 [48]). *Let $a, b \in \mathbb{Z}^+$, $a, b \geq 2$. If $R(a, b-1)$ and $R(a-1, b)$ both exist, then*

$$R(a, b) \leq R(a, b-1) + R(a-1, b).$$

Proof. Assume $R(a, b-1)$ and $R(a-1, b)$ exist. Let $n = R(a, b-1) + R(a-1, b)$. It suffices to show that

$$K_n \longrightarrow (K_a, K_b)_2^{K_2} \tag{2.1}$$

Let $\Delta : E(K_n) \to \{\text{red}, \text{blue}\}$, and fix some $v \in V(K_n)$. Let X and Y be the sets of vertices attached to v by red and blue edges respectively (see Figure 2.2).

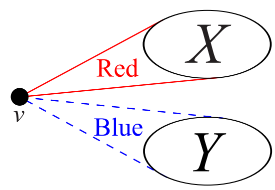


Figure 2.2: Proving the Erdős-Szekeres recursion

If $|X| \geq R(a-1, b)$, then X either contains a blue K_b, in which case (2.1) is satisfied, or a red K_{a-1}, which with v forms a red K_a, and (2.1) is again satisfied. So assume $|X| < R(a-1, b)$. Then

$$|Y| = n - 1 - |X| = R(a, b-1) + R(a-1, b) - 1 - |X| > R(a, b-1) - 1,$$

and therefore $|Y| \geq R(a, b-1)$. So, Y contains either a red K_a, and (2.1) is satisfied, or a blue K_{b-1}, which with v forms a blue K_b, again satisfying (2.1). Therefore $K_n \longrightarrow (K_a, K_b)_2^{K_2}$. □

Proof of $k = 2$ case of Ramsey's theorem (Due to Erdős and Szekeres, 1935 [48]). The proof is by induction on $a+b$. For any $n \in \mathbb{Z}^+$, $n \geq 4$, let $S(n)$ denote the statement that for all $a, b \in \mathbb{Z}^+$ such that $a + b = n$, $R(a, b)$ exists. If for all $n \geq 4$, $S(n)$ holds, then since for any $m \geq 2$, $R_2(m; 2) = R(m, m)$, Ramsey's theorem (for $k = 2$) would hold as well.

Base case: When $\min\{a, b\} < 2$, the theorem is a triviality. When $\min\{a, b\} = 2$ (which includes the case $a + b = 4$), $R(a, b) = \max\{a, b\}$. Therefore $S(4)$ holds.

Inductive Step: Let $n \geq 4$, and assume that $S(n)$ holds. Let $a, b \geq 2$ be such that $a + b = n + 1$. Then by $S(n)$, both $R(a-1, b)$ and $R(a, b-1)$ exist, and by the Erdős-Szekeres recursion,

$$R(a, b) \leq R(a-1, b) + R(a, b-1).$$

Therefore $R(a, b)$ exists, and $S(n+1)$ holds.

By mathematical induction, for all $n \geq 4$, $S(n)$ holds, which proves the theorem. □

2.7.2 Colouring larger complete graphs

The Erdős-Szekeres recursion was originally published in the following more general version.

Theorem 2.7.2 (Erdős-Szekeres recursion (general version), 1935 [48])**.** *Let $a, b, k \in \mathbb{Z}^+$. If $R_{k-1}(R_k(a-1,b), R_k(a,b-1))$ exists, then*

$$R_k(a,b) \leq R_{k-1}(R_k(a-1,b), R_k(a,b-1)) + 1.$$

Proof. Assume that $R_{k-1}(R_k(a-1,b), R_k(a,b-1))$ exists. Set $p = R_k(a-1,b)$, $q = R_k(a,b-1)$, $n = R_{k-1}(p,q) + 1$, and let X be an n-set. Let $\Delta : [X]^k \to \{\text{red}, \text{blue}\}$, fix $v \in X$, and define a colouring $\Delta' : [X \setminus \{v\}]^{k-1} \to \{\text{red}, \text{blue}\}$ by $\Delta'(T) = \Delta(T \cup \{v\})$. By the definition of n, either (a) there exists $Y \in [X \setminus \{v\}]^p$ such that $[Y]^{k-1}$ is monochromatic red under Δ', or (b) there exists $Z \in [X \setminus \{v\}]^q$ such that $[Z]^{k-1}$ is monochromatic blue under Δ'.

If (a) holds, then by definition of p, it can be assumed that there exists $Y' \in [Y]^{a-1}$ such that $[Y']^k$ is monochromatic under Δ. But then $Y' \cup \{v\}$ is an a-set such that $[Y' \cup \{v\}]^k$ is monochromatic red under Δ. The same argument works if (b) holds, except replacing Y, p, and a by Z, q, and b respectively. $\square$

Third Proof of Ramsey's theorem (Due to Erdős and Szekeres, 1935 [48]). The proof is by induction on k and $a+b$. For any fixed $k, m \in \mathbb{Z}^+$, $m \geq 4$, let $S(k,m)$ be the statement that for all $a, b \geq 2$ such that $a + b = m$, $R_k(a, b)$ exists. As before, if for any $m \geq 4$ and $k \in \mathbb{Z}^+$, $S(k, m)$ holds, then since $R_k(m; 2) = R_k(m, m)$, Ramsey's theorem would also hold.

Base Cases: By the pigeonhole principle, for all $a, b \geq 2$, $R_1(a, b) = a + b - 1$, and so for all $m \geq 4$, $S(1, m)$ holds. For all $k \in \mathbb{Z}^+$, if $\min\{a, b\} \leq k$, then $R_k(a, b) = \min\{a, b\}$. Therefore for all $k, m \in \mathbb{Z}^+$ such that $m \leq 2k$, $S(k, m)$ holds.

Inductive Step: Let $k \geq 1$, $m > \max\{2k, 5\}$, and assume that

$$S(k, m-1) \text{ holds, and} \tag{2.2}$$

$$\text{for all } m' \geq 4,\ S(k-1, m') \text{ holds.} \tag{2.3}$$

Let $a, b \geq 2$ be such that $a + b = m$. Then by Theorem 2.7.2,

$$R_k(a, b) \leq R_{k-1}(R_k(a-1, b), R_k(a, b-1)) + 1. \tag{2.4}$$

By (2.2), $R_k(a-1, b)$ and $R_k(a, b-1)$ both exist, and thus by (2.3), the right hand side of (2.4) exists. Therefore $R_k(a, b)$ exists, and thus $S(k, m)$ holds.

Thus by induction, for all $k, m \in \mathbb{Z}^+$, $m \geq 4$, $S(k, m)$ holds, which proves the theorem. $\square$

2.8 The n-gon problem

In 1935, Ramsey's theorem was applied by Erdős and Szekeres [48] in a geometric setting, which, according to a number of authors (see, *e.g.*, [65, p. 25] and [121]) helped Ramsey's theorem grow in popularity among mathematicians in fields other than logic.

For purposes here, $A \subseteq \mathbb{R}^2$ is a *polygon* iff A is connected, is bounded by a closed sequence of finitely many straight line segments, and A contains its boundary. For this discussion, a polygon also contains its interior. A polygon A is said to be *convex* iff for every $x, y \in A$, the line segment connecting x and y is contained entirely within A. That is, A is convex iff for every $x, y \in A$ and every $\lambda \in \mathbb{R}$ with $0 \leq \lambda \leq 1$, $\lambda x + (1 - \lambda)y \in A$. Given a set of points P, the *convex hull* of P is the intersection of all convex sets containing every point in P, and so is the smallest polygon containing every point in P. A set of points is said to be in *general position* iff no three points are collinear. E. Klein proposed and proved the following proposition (as recorded in Erdős and Szekeres' paper [48]).

Proposition 2.8.1. *Given any five points in the plane in general position, some four form a convex quadrilateral (that is, the convex hull of some four is a convex quadrilateral).*

Proof. Since there are three non-collinear points, the convex hull of any five points is a polygon with at least three sides. There are three cases, as shown in Figure 2.3. If the convex hull is either a quadrilateral or a pentagon, there is nothing to prove.

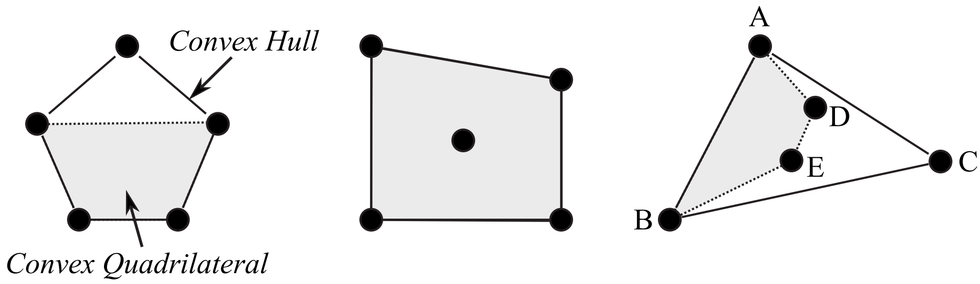


Figure 2.3: Any five points in general position contain a convex quadrilateral

Otherwise, the convex hull is a triangle. Let A, B, C be the vertices of this triangle, and let D and E be the two other points (contained in the interior of the triangle). Then two vertices of the triangle, say A and B, are on the same side of the line $\overline{DE}$, and therefore there is a convex quadrilateral on the vertices A, B, D, and E. $\square$

According to [48], the following was suggested by Klein, and proved in the affirmative by Erdős and Szekeres.

Theorem 2.8.2 (Erdős and Szekeres, 1935 [48])**.** *For any $m \in \mathbb{Z}^+$, there exists a least integer $n = ES(m)$ such that for any n points in the plane in general position, some m points form a convex m-gon.*

Four proofs are presented here, the first of which is a direct proof, while the other three use Ramsey's theorem.

First proof of Theorem 2.8.2 (Due to Erdős and Szekeres, 1935 [48]). For two points in $\mathbb{R}^2$ $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ where $x_1 \neq x_2$, let $m(P_1, P_2)$ denote the slope of the line through P_1 and P_2, *i.e.*,

$$m(P_1, P_2) = \frac{y_2 - y_1}{x_2 - x_1}.$$

Let $C = \{P_1, \ldots, P_k\}$ be a set of points in $\mathbb{R}^2$, sorted by ascending x-coordinates (without loss of generality, by tilting, it may be assumed that the x-coordinates of the points in C are all distinct). If for each $i \in [1, k-2]$,

$$m(P_i, P_{i+1}) > m(P_{i+1}, P_{i+2}),$$

then C is said to be a k-*cap*. If for all $i \in [1, k-2]$,

$$m(P_i, P_{i+1}) < m(P_{i+1}, P_{i+2}),$$

then C is said to be a k-*cup*. Let $f(i, k)$ denote the least $n \in \mathbb{Z}^+$ (if any exists) such that given any n points in general position, some i of them form an i-cup, or some k of them form a k-cap. If for some $m \in \mathbb{Z}^+$, $f(m, m)$ exists, then since any cup or cap can be made into a convex polygon by joining the first and last points, it follows that $ES(m) \leq f(m, m)$.

Claim. For all $i, j \in \mathbb{Z}^+$, $i, j \geq 4$, $f(i, j) \leq f(i-1, j) + f(i, j-1) - 1$.

Proof of Claim. Let $i, j \geq 4$, assume $f(i-1, j)$ and $f(i, j-1)$ exist, and let $n = f(i-1, j) + f(i, j-1) - 1$. Let $p_1, \ldots, p_n$ be any n points in $\mathbb{R}^2$, ordered by ascending

x-coordinates. In an effort to prove the claim, a set of points $\{v_1, \ldots, v_{f(i,j-1)}\} \subseteq \{p_1, \ldots, p_n\}$ is first constructed inductively.

If the points $p_1, \ldots, p_{f(i-1,j)}$ do not contain some j points that form a j-cap, then there exists $\beta_1^1 \leq \ldots \leq \beta_{i-1}^1 \in [1, f(i-1,j)]$ such that $p_{\beta_1^1}, \ldots, p_{\beta_{i-1}^1}$ form an $(i-1)$-cup. Let $v_1 = p_{\beta_{i-1}^1}$.

Let $k \in [1, f(i,j-1)-1]$, and assume that v_k has already been defined. Assume the set of points $\{p_1, \ldots, p_{f(i-1,j)+k}\} \setminus \{v_1, \ldots, v_k\}$ does not contain some j points that form a j-cap. Then since

$$|\{p_1, \ldots, p_{f(i-1,j)+k}\} \setminus \{v_1, \ldots, v_k\}| = f(i-1,j),$$

there exists $\beta_1^{k+1} \leq \ldots \leq \beta_{i-1}^{k+1}$ such that $p_{\beta_1^{k+1}}, \ldots, p_{\beta_{i-1}^{k+1}}$ form an $(i-1)$-cup. Let $v_{k+1} = p_{\beta_{i-1}^{k+1}}$.

Having defined the points $\{v_1, \ldots, v_{f(i,j-1)}\}$, each an end point of some $(i-1)$-cup, assume that no i points in $\{v_1, \ldots, v_{f(i,j-1)}\}$ produce an i-cup. Then there exist $\{r_1, \ldots, r_{j-1}\} \subseteq \{v_1, \ldots, v_{f(i,j-1)}\}$ that form a $(j-1)$-cap. Let $\{q_1, \ldots, q_{i-1}\}$ be an $(i-1)$-cup such that $q_{i-1} = r_1$ (see Figure 2.4). If the slope from r_1 to r_2 is greater than the slope from q_{i-2} to r_1, then $q_1, \ldots, q_{i-2}, q_{i-1}, r_2$ is an i-cup. Otherwise $q_{i-2}, q_{i-1}, r_2, \ldots, r_{j-1}$ is a j-cap. This proves the claim.

A simple inductive argument using the base case $f(3,m) = f(m,3) = m$ now

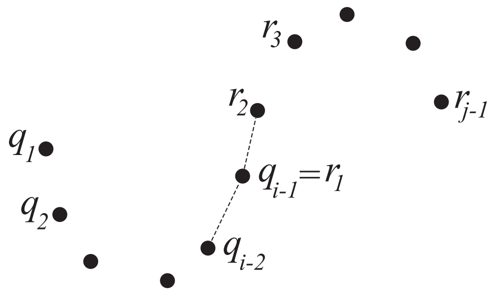


Figure 2.4: The points $q_1, \ldots, q_{i-1}$ and $r_1, \ldots, r_{j-1}$

shows that for all $i, j \in \mathbb{Z}^+$, $i, j \geq 3$, $f(i, j)$ exists and is finite. Therefore, since $ES(m) \leq f(m, m)$, $ES(m)$ exists and is finite for all $m \in \mathbb{Z}^+$. $\square$

Second proof of Theorem 2.8.2 (Due to Erdős and Szekeres, 1935 [48]). Fix $m \in \mathbb{Z}^+$, and let $n = R_4(5, m)$. Fix any n points in the plane in general position, and let Δ be a 2-colouring of the four element subsets of points defined by $\Delta(\{a, b, c, d\}) =$ blue if $\{a, b, c, d\}$ forms a convex quadrilateral, $\Delta(\{a, b, c, d\}) =$ red otherwise. Consider any five points among the n. Among these five points, by Proposition 2.8.1, some four form a convex quadrilateral, and therefore some four are coloured blue. Thus, there can be no set of five points with all 4-element subsets coloured red. Then, by the choice of n, there are m points $p_1, \ldots, p_m$, every four of which form a convex quadrilateral.

In hopes of a contradiction, assume that $p_1, \ldots, p_m$ do not form a convex m-gon. Then one point, call it p^*, would be contained in the interior of the convex hull of these m points, and therefore, within the interior of the convex hull of some three points, call them q, r, and s. Then p^*, q, r, and s do not form a convex quadrilateral, a contradiction. □

A proof by Michael Tarsy was presented by M. Lewin in 1976 [90]. Lewin writes that Tarsy, an undergraduate student at Technion Israel Institute of Technology in Lewin's combinatorics class, was asked to prove the Erdős-Szekeres n-gon theorem on his exam. The class had been shown the above proof due to Erdős and Szekeres (using $R_4(5, m)$), but Tarsy was "fortunately" [65, p. 26] absent from class that day.

Third proof of Theorem 2.8.2 (Due to Tarsy, see [90, p. 136]). Let $m \geq 2$, let $n = R_3(m; 2)$, and let $V = \{v_1, \ldots, v_n\}$ be a set of n points in the plane in general position. For $i < j < k \in [1, n]$, let (v_i, v_j, v_k) denote the triangle with vertices v_i, v_j, and v_k. Call the triangle (v_i, v_j, v_k) a counter clockwise (ccw) triangle if v_i, v_j, and v_k occur in counterclockwise orientation, and a clockwise (cw) triangle otherwise. Define $\Delta : [V]^3 \to \{\text{red}, \text{blue}\}$ by colouring the triple $\{v_i, v_j, v_k\}$ red if (v_i, v_j, v_k) is a ccw triangle, and blue otherwise.

By Ramsey's theorem, there exists $M \in [V]^m$ such that each triangle in M has the same orientation, say all triangles are ccw. Assume these m points do not form a convex m-gon. Then for some $a, b, c, d \in [1, n]$, v_d lies inside the triangle (v_a, v_b, v_c).

Without loss of generality, assume that $a < b < c$.

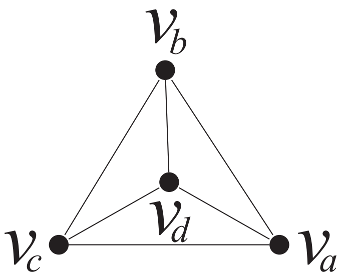


Figure 2.5: The ccw triangle (v_a, v_b, v_c) containing v_d.

In order that the points v_a, v_c, and v_d form a ccw triangle, either (i) $a < d < c$, or (ii) $d < c < a$ or (iii) $c < a < d$. Neither (ii) nor (iii) can hold since $a < c$, and therefore $a < d < c$. In the same way, in order that v_a, v_b and v_d form a ccw triangle, either (iv) $a < b < d$, (v) $b < d < a$, or (vi) $d < a < b$. By (i), $a < d$, and so neither (v) or (vi) hold. Therefore $a < b < d$. However, $b < d$ implies that $a < b < d < c$, implying that (v_b, v_d, v_c) is a ccw triangle, which is not true.

Therefore the points in M form a convex m-gon. $\square$

Fourth proof of Theorem 2.8.2 (Due to Johnson [78]). Let $m \in \mathbb{Z}^+$, let $n = R_3(m)$, and let V be a set of n points in the plane in general position. For any $a, b, c \in V$, let $N(a, b, c)$ denote the number of elements of V inside the triangle formed by vertices a, b, and c. Define $\Delta : [V]^3 \to \{\text{red}, \text{blue}\}$ by

$$\Delta(\{a,b,c\}) = \begin{cases} \text{red} & \text{if } N(a,b,c) \text{ is even,} \\ \text{blue} & \text{if } N(a,b,c) \text{ is odd.} \end{cases}$$

Then by the choice of n, there exists an m-set M such that Δ is constant on $[M]^3$. If the points of M did not form a convex m-gon, there would exist $a, b, c, d \in M$ such that d is inside the triangle formed by a, b, c, in which case

$$N(a,b,c) = N(a,b,d) + N(a,c,d) + N(b,c,d) + 1. \tag{2.5}$$

Since Δ is constant on $[M]^3$, $N(a,b,d)$, $N(a,c,d)$, $N(b,c,d)$, and $N(a,b,c)$ are all the same parity, an impossibility by equation (2.5).[1] □

Corollary 2.8.3. *For all* $m \in \mathbb{Z}^+$, $ES(m) \leq \min\{R_3(m), R_4(m,5)\}$.

2.9 The Paris-Harrington theorem

The following theorem of J. Paris and L. Harrington generalizes Ramsey's theorem, and is notable as a statement true in Peano arithmetic, but is not provable within Peano arithmetic (see [65, p. 170] for details). Much of the proof given here

[1]The last three proofs show that Ramsey's theorem implies the Erdős-Szekeres n-gon theorem. It has been suggested that the Erdős-Szekeres n-gon theorem implies Ramsey's theorem (at least in the $k = 2$ case), though at present I have been unable to find a proof of this fact.

duplicates the second proof of Ramsey's theorem (see Section 2.5). Define a set $S \subseteq \mathbb{Z}^+$ to be *large* iff $|S| \geq \min S$. For example, $\{3, 6, 7, 10\}$ is large, but $\{9, 15, 22, 34\}$ is not.

Theorem 2.9.1 (Paris and Harrington, 1977 [116])**.** *Let $k, r, s \in \mathbb{Z}^+$. Then there exists $n \in \mathbb{Z}^+$ such that for any $\Delta : [1, n]^k \to [1, r]$, there exists a large $S \subseteq [1, n]$ with $|S| \geq s$ such that Δ is constant on $[S]^k$.*

Proof. In hopes of a contradiction, assume for all finite n, there exists an r-colouring $\Delta : [1, n]^k \to [1, r]$ such that every large set of order at least s is not monochromatic under Δ. Call such a colouring a *bad colouring of* $[1, n]^k$.

Let $\Delta : [\mathbb{Z}^+]^k \to [1, r]$. By Ramsey's theorem for infinite sets (Theorem 1.2.4), there exists $T \subseteq \mathbb{Z}^+$ such that $|T| = |\mathbb{Z}^+|$, and Δ is constant on $[T]^k$. Let $n = \max\{s, \min T\}$, and let S be the first n elements of T. Then S is a large set, monochromatic under Δ, and therefore there is no bad colouring of $[\mathbb{Z}^+]^k$.

Let

$$V = \{\Delta : \Delta \text{ is a bad colouring of } [1, n]^k,\ n \in \mathbb{Z}^+ \cup \{0\}\}.$$

Note that if Δ is a bad colouring of $[1, n]^k$, then $\Delta|_{[1,n-1]^k}$ is also a bad colouring. Note also that the empty colouring is a bad colouring.

Let $E \subseteq [V]^2$ be defined as follows: for $i \leq j$, a bad colouring Δ_i of $[1, i]^k$, and a bad colouring Δ_j of $[1, j]^k$, let $\{\Delta_i, \Delta_j\} \in E$ iff $j = i + 1$ and $\Delta_j|_{[1,i]^k} = \Delta_i$. Then

(V, E) is an infinite locally finite tree, rooted at the empty colouring. By König's infinity lemma (Theorem 2.5.1), there exists an infinite path $\Delta_0, \Delta_1, \ldots$ starting at the root. This infinite path corresponds to a bad colouring $\Delta_\infty : [\mathbb{Z}^+]^k \to [1, r]$, which extends each colouring $\Delta_0, \Delta_1, \ldots$, but this was shown to be impossible. $\square$

Chapter 3

Traditional Ramsey numbers

Recall that (see Theorem 1.2.3) $R_k(m;r)$ is the least integer n such that for every n-set N, and for every r-colouring $\Delta : [N]^k \to [1,r]$, there exists $M \in [N]^m$ such that $[M]^k$ is monochromatic under Δ. In Chapter 2, the off-diagonal generalization $R_k(a,b)$ was also defined, and both $R_k(m;r)$ and $R_k(a,b)$ were shown to exist. The focus of this chapter is to survey results on bounding $R_k(m;r)$ and $R_k(a,b)$.

3.1 Exact Values

Before presenting some of the few known non-trivial values for the numbers $R_k(m;r)$, and $R_k(a,b)$, some trivial cases are exhibited.

3.1.1 Trivial cases

There are a number of trivial values and bounds that can be determined immediately using the definition of the Ramsey numbers. The following two (trivial) values are direct consequences of the pigeonhole principle.

Observation 3.1.1. *For all* $m, r \in \mathbb{Z}^+$, $R_1(m; r) = (m-1)r + 1$.

Observation 3.1.2. *For all* $a, b \in \mathbb{Z}^+$, $R_1(a, b) = a + b - 1$.

The next two cases cover some vacuous cases. When $k > m$, $R_k(m; r)$ is defined as the least integer n so that for every n-set N, and every $\Delta : [N]^k \to [1, r]$, there exists $M \in [N]^m$ such that $[M]^k$ $(= \emptyset)$ is monochromatic under Δ. We say that, vacuously, every $M \in [N]^m$ is such that $[M]^k$ is monochromatic under Δ, and therefore $R_k(m; r) = m$. The following observations sum up these vacuous cases, in both the diagonal and off-diagonal cases.

Observation 3.1.3. *For* $k, m, r \in \mathbb{Z}^+$, *if* $k \geq m$, *then* $R_k(m; r) = m$

Observation 3.1.4. *For* $a, b, k \in \mathbb{Z}^+$, *if* $k > \min\{a, b\}$, *then* $R_k(a, b) = \min\{a, b\}$, *and if* $k = \min\{a, b\}$, *then* $R_k(a, b) = \max\{a, b\}$.

In order to avoid the above trivial situations, one may assume that from now on, $m > k$.

Note that the off-diagonal Ramsey numbers are symmetric.

Lemma 3.1.5. *For all $a, b, k \in \mathbb{Z}^+$, $R_k(a, b) = R_k(b, a)$.*

Proof. Let $a, b, k \in \mathbb{Z}^+$. It suffices by symmetry to show that $R_k(a, b) \leq R_k(b, a)$. Let $n = R_k(b, a)$, let N be an n-set, and let $\Delta : [N]^k \to \{\text{red}, \text{blue}\}$. Define $\Delta' : [N]^k \to \{\text{blue}, \text{red}\}$ by $\Delta'(X) = \text{red}$ if $\Delta(X) = \text{blue}$, and $\Delta'(X) = \text{blue}$ otherwise. By the definition of n, there is either a set $M_1 \in [N]^b$ monochromatic blue under Δ' (and therefore monochromatic red under Δ), or there is a set $M_2 \in [N]^a$ monochromatic red under Δ' (and therefore monochromatic blue under Δ). $\square$

All the following bounds are in the $k = 2$ case. Recall that $R(a, b) = R_2(a, b)$. An *independent set* of vertices in a graph is a set of vertices containing no edge. To show that $R(a, b) > k$, it suffices to exhibit a 2-colouring of the edges of K_k containing no red K_a and no blue K_b, or, equivalently, to exhibit a K_a-free graph G on k vertices such that there is no independent set with b vertices.

To show that $R(a, b) \leq \ell$, it suffices to prove that for every 2-colouring of the edges of K_ℓ, there is either a monochromatic red K_a or a monochromatic blue K_b, or, equivalently, to prove that every K_a-free graph on ℓ vertices must contain an independent set of order b.

3.1.2 The party problem

The following problem is now folklore in graph Ramsey theory.

The Party Problem. *Prove that at a party of six people, either there are three mutual acquaintances or there are three mutual strangers.*

The party problem was on the 13th William Lowell Putnam Mathematical Competition in 1953, and in *The American Mathematical Monthly*, in 1958 [50]. Assuming that "knowing" people is a symmetric relation (where in real life it's not), solving the party problem is equivalent to the graph Ramsey theory problem of showing that no matter how the edges of K_6 are coloured with two colours (colouring an edge red, say, if two people are acquaintances, and blue if they are strangers), one colour class must contain a triangle, *i.e.*, $R(3,3) \leq 6$. In fact, $R(3,3) = 6$.

Theorem 3.1.6. $R(3,3) = 6$.

Proof. To show that $R(3,3) > 5$, let Δ be the colouring of $E(K_5)$ shown in Figure 3.1 (dashed lines represent edges coloured blue and solid lines represent edges coloured red). Every triangle (copy of K_3) in K_5 under Δ contains two edges that are different colours. Therefore $K_5 \not\rightarrow (K_3)_2^{K_2}$, showing $R(3,3) > 5$.

To show that $R(3,3) \leq 6$, let $\Delta : E(K_6) \rightarrow \{\text{red}, \text{blue}\}$. Let $V(K_6) = \{v, a, b, c, x, y\}$. By the pigeonhole principle, at least three of the five edges attached to v, say $\{v, a\}$, $\{v, b\}$, and $\{v, c\}$, are coloured the same. Without loss of generality, assume these

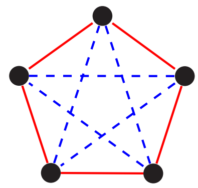


Figure 3.1: A colouring of $E(K_5)$ containing no monochromatic triangle.

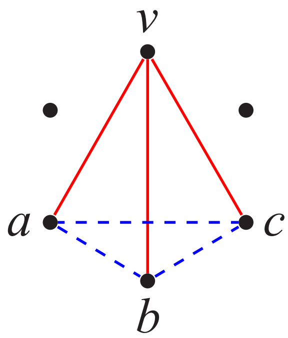


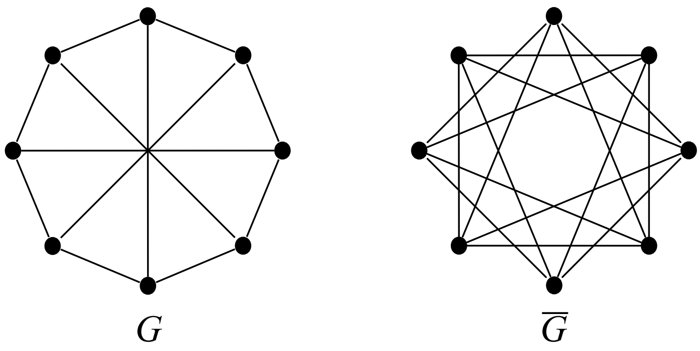
Figure 3.2: The party problem

three edges are all coloured red. If any one of $\{a, b\}$, $\{b, c\}$, or $\{a, c\}$ is coloured red, then a red triangle is formed. Otherwise, $\{a, b\}$, $\{b, c\}$, and $\{a, c\}$ are all blue, and together they form a blue triangle (see Figure 3.2). Therefore $K_6 \longrightarrow (K_3)_2^{K_2}$, showing that $R(3, 3) \leq 6$. $\square$

3.1.3 R(3,4) = 9

Theorem 3.1.7 (Greenwood and Gleason, 1955 [67]). $R(3, 4) = 9$.

Proof. To see why $R(3, 4) > 8$, consider the following graph G and its complement:



By inspection one can verify that there are no triangles in G and that there are no K_4's in $\overline{G}$. Hence $R(3,4) > 8$.

To see why $R(3,4) \leq 9$, let $\Delta : E(K_9) \to \{\text{red}, \text{blue}\}$. For any $x \in V(K_9)$, let A_x be the set of vertices connected to x by red edges, and let B_x be those vertices connected to x by blue edges.

If for any $x \in V(K_9)$, $|B_x| \geq 6$, then since $R(3,3) = 6$, B_x must contain either a red triangle (in which case the theorem is proved), or a blue triangle, which together with x forms a blue K_4 (and the theorem is again proved). Therefore assume that for every $x \in V(K_9)$, $|B_x| \leq 5$.

If for any $x \in V(K_9)$, $|A_x| \geq 4$, then since $R(2,4) = 4$ (a trivial case covered by Observation 3.1.4), A_x either contains a red edge, which together with x forms a red triangle (and the theorem is proved), or contains a blue K_4 (and the theorem is again proved). Therefore assume that for every $x \in V(K_9)$, $|A_x| \leq 3$.

Since for every $x \in V(K_9)$, $|B_x| \leq 5$, and $|A_x| \leq 3$, and $|A_x| + |B_x| = 8$, it follows that $|A_x| = 3$ and $|B_x| = 5$. Let G be the subgraph of K_9 on the red edges. Then G is 3-regular, and by the handshaking lemma (Lemma A.0.4),

$$2|E(G)| = \sum_{v \in V(G)} deg(v) = 3|V(G)| = 27,$$

which is impossible, since $|E(G)|$ is an integer. Therefore it cannot happen that every vertex of K_9 is on exactly three red edges and five blue edges. This proves the theorem. $\square$

3.1.4 R(4,4) = 18

Theorem 3.1.8 (Greenwood and Gleason, 1955 [67]). $R(4,4) = 18$.

Proof. By the Erdős-Szekeres recursion (Theorem 2.7.1),

$$R(4,4) \leq R(3,4) + R(4,3) = 9 + 9 = 18.$$

To show that $R(4,4) > 17$, first recall that for a prime p, an integer $x \in [1, p-1]$, $x \neq 0$, is a *quadratic residue (mod p)* if and only if there exists y such that $y^2 = x$ (mod p). Let G be the graph with vertices $\{0, 1, \ldots, 16\}$, and edge set $\{\{i, j\} : i - j \text{ is a quadratic residue (mod 17)}\}$ (see Figure 3.3). Note that the quadratic residues modulo 17 are $\{1, 2, 4, 8, 9, 13, 15, 16\}$.

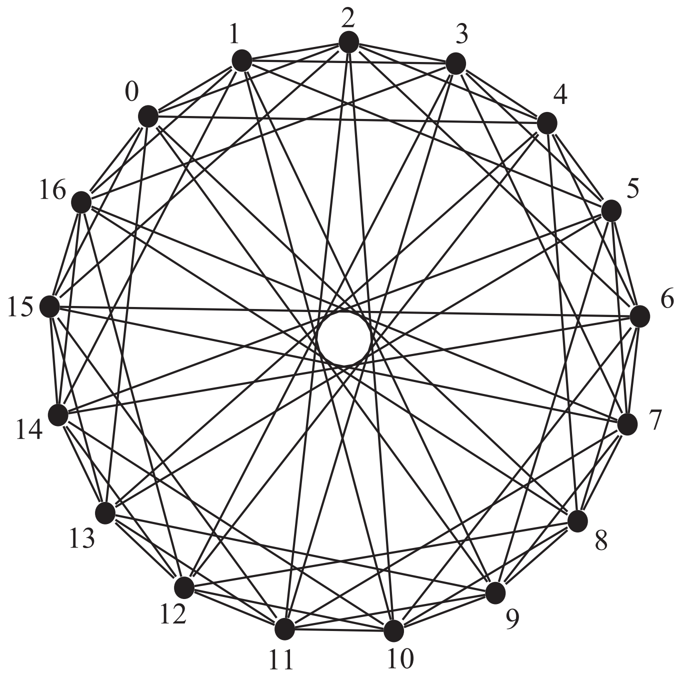


Figure 3.3: A graph G showing $R(4,4) > 17$.

Claim. The graph G contains no K_4.

Proof. Let $e = \{i, j\}$, $i < j$, be any (fixed) edge in G. Note that the function $\psi : x \mapsto x - i$ is an isomorphism between G and itself (since $\{a, b\} \in E(G)$ iff $a - b$ is a quadratic residue (mod 17) iff $(a - i) - (b - i)$ is a quadratic residue (mod 17) iff $\{a - i, b - i\} \in E(G)$). Therefore, it can be assumed that $i = 0$. Similarly, for any two quadratic residues x_1 and x_2, $\frac{x_1}{x_2}$ is also a quadratic residue, and therefore it can also be assumed that $j = 1$.

Consider two triangles in G sharing the edge $\{0, 1\}$, say $0, 1, k$ and $0, 1, j$. Then k and $k - 1$ are both quadratic residues mod 17, and thus $k \in \{2, 9, 16\}$. Similarly,

$j \in \{2, 9, 16\}$. Since none of $\{2, 9\}$, $\{9, 16\}$, or $\{2, 16\}$ are edges in G, these two triangles cannot be contained in some K_4. Therefore G cannot contain any K_4's.

To see that $\overline{G}$ doesn't contain any K_4's either, notice that $\overline{G}$ is isomorphic to G, as exhibited by[1] the isomorphism $f : x \mapsto 3x \pmod{17}$. $\square$

The graph G used above is an example of a "Paley graph" (see, *e.g.*, [8]). For any prime power q, $q \equiv 1 \pmod 4$, the *Paley graph of order* q, denoted P_q, is the graph with vertex set $[0, q-1]$ and $\{x, y\} \in E(P_q)$ iff $x - y$ is a quadratic residue (mod q). The requirement that $q \equiv 1 \pmod 4$ ensures that -1 is a square, making $i - j$ a quadratic residue iff $j - i$ is a quadratic residue, and thus the resulting graph is undirected. The properties of Paley graphs used above are that Paley graphs are *edge-transitive*, that is, for any two edges $e_1 = \{x_1, y_1\}, e_2 = \{x_2, y_2\} \in E(P_q)$, there exists an automorphism ϕ_{e_1,e_2} of P_q such that $\phi(x_1) = x_2$ and $\phi(y_1) = y_2$, and that Paley graphs are self-complimentary, that is, $P_q \cong \overline{P_q}$.

[1] thanks to K. Johannson for showing me this

3.1.5 Summary of known exact values

a \ b	3	4	5	6	7	8	9
3	6	9	14	18	23	28	36
4		18	25				

Table 3.1: Known exact Ramsey numbers $R(a, b)$.

The values in Table 3.1 are the only exact non-trivial values known for the traditional Ramsey numbers $R(a, b)$. A number of bounds are known for the other values $R(a, b)$, *e.g.*, some of the best-known bounds on the small diagonal numbers $R(a, a)$ include the following:

$$43 \leq R(5,5) \leq 49 \qquad 102 \leq R(6,6) \leq 165$$

$$205 \leq R(7,7) \leq 540 \qquad 282 \leq R(8,8) \leq 1870$$

$$565 \leq R(9,9) \leq 6588 \qquad 798 \leq R(10,10) \leq 23556$$

For references and known bounds, see the dynamic survey by Radziszowski [123].

The arrow notation, and the Ramsey numbers $R_k(a, b)$ have a natural generalization to r colours. For $k, r, m_1, \ldots, m_r \in \mathbb{Z}^+$, write $n \longrightarrow (m_1, \ldots, m_r)_r^k$ if for any n-set N, and for any $\Delta : [N]^k \to [1, r]$, there exists $i \in [1, r]$, and $M_i \in [N]^{m_i}$ such that $[M_i]^k$ is monochromatic in the i-th colour. Let $R_k(m_1, \ldots, m_r)$ be the least integer

n such that

$$n \to (m_1, \ldots, m_r)^k_r.$$

The values $R_k(m_1, \ldots, m_r)$ do exist since

$$R_k(m_1, \ldots, m_r) \leq R_k(\max\{m_1, \ldots, m_r\}; r),$$

but other than trivial values similar to those in Observations 3.1.1 through 3.1.4, the only value known is $R_2(3, 3, 3) = R_2(3; 3) = 17$ (due to Greenwood and Gleason [67]).

3.2 Erdős-Szekeres upper bound

The Erdős-Szekeres recursion given in Theorem 2.7.2 produces an upper bound on the Ramsey numbers by induction.

Theorem 3.2.1 (Erdős and Szekeres, 1935 [48]). *For all $k, \ell \geq 2$, $R(k, \ell) \leq \binom{k+\ell-2}{k-1}$.*

Proof. The proof of Theorem 3.2.1 is by induction on $k+\ell$. For any $n \in \mathbb{Z}^+$, $n \geq 4$, let $S(n)$ be the statement that for all $k, \ell \geq 2$ such that $k+\ell = n$, $R(k, \ell) \leq \binom{k+\ell-2}{k-1}$.

Base Case: When $k = \ell = 2$, the theorem says that $R(2, 2) \leq \binom{2+2-2}{2-1} = \binom{2}{1} = 2$, which is true. Therefore $S(4)$ holds.

Inductive Step: Let $m \geq 4$ and assume that $S(m)$ holds. Let $k, \ell \geq 2$ be such that

$k + \ell = m + 1$. It remains to show that $R(k, \ell) \leq \binom{k+\ell-2}{k-1}$.

When $k = 2$, the theorem says that $R(2, \ell) \leq \binom{2+\ell-2}{2-1} = \binom{\ell}{1} = \ell$ which is true. Therefore the theorem holds whenever $k = 2$ (and, by symmetry, whenever $\ell = 2$). If $k, \ell \geq 3$, then

$$\begin{aligned} R(k,\ell) &\leq R(k-1,\ell) + R(k,\ell-1) && \text{(by the Erdős-Szekeres recursion)}, \\ &\leq \binom{k+\ell-3}{k-2} + \binom{k+\ell-3}{k-1} && \text{(since } S(m) \text{ holds)}, \\ &= \binom{k+\ell-2}{k-1} && \text{(by Pascal's Equality)}. \end{aligned}$$

Therefore $S(m + 1)$ holds.

By induction, for all $n \geq 4$, $S(n)$ holds, proving the theorem. □

In order to compute a closed upper bound on the Ramsey numbers, Stirling's formula is used. The approximation for $n!$ given by Stirling (see *e.g.*, [143, p. 384]) is for any $n \in \mathbb{Z}^+$,

$$\sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\frac{1}{12n+1}} < n! < \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\frac{1}{12n}}.$$

Corollary 3.2.2. *For all $k \geq 2$,*

$$R(k, k) \leq \frac{1}{4\sqrt{k}}\, 2^{2k}.$$

Proof. Let $k \geq 2$. Then,

$$R(k,k) \leq \binom{2k-2}{k-1} \qquad \text{(by Theorem 3.2.1)}$$

$$\begin{aligned}
&= \frac{(2k-2)!}{(k-1)!(k-1)!} \\
&< \sqrt{2\pi(2k-2)}\left(\frac{2k-2}{e}\right)^{2k-2} e^{\frac{1}{12(2k-2)}}\left(\frac{e}{k-1}\right)^{2(k-1)} \frac{1}{2\pi(k-1)} e^{\frac{2}{12(k-1)+1}} \\
&= \left(\frac{2k-2}{k-1}\right)^{2k-2}\left(\frac{1}{\sqrt{2\pi}}\right)\left(\frac{\sqrt{2k-2}}{k-1}\right) e^{\frac{1}{12(2k-2)}} e^{\frac{2}{12(k-1)+1}} \\
&= 2^{2k-2}\left(\frac{1}{\sqrt{\pi}}\right)\left(\frac{\sqrt{k-1}}{k-1}\right) e^{\frac{1}{12(2k-2)}} e^{\frac{2}{12(k-1)+1}} \\
&= 2^{2k}\left(\frac{1}{4\sqrt{\pi}}\right)\left(\frac{1}{\sqrt{k-1}}\right) e^{\frac{1}{12(2k-2)}} e^{\frac{2}{12(k-1)+1}} \\
&= 2^{2k}\left(\frac{1}{4\sqrt{\pi}}\right)\left(\frac{1}{\sqrt{k}}\right)\left(\sqrt{\frac{k}{k-1}}\right) e^{\frac{1}{12(2k-2)}} e^{\frac{2}{12(k-1)+1}} \\
&< 2^{2k}\left(\frac{1}{4\sqrt{k}}\right) \qquad \square
\end{aligned}$$

Corollary 3.2.2 has been improved a number of times. In 1986, Rödl presented (but apparently didn't publish—see, *e.g.*, [139, p. 509] and [102, p. 1348]) the result that there exists a constant c such that

$$R(k,\ell) \leq \frac{\binom{k+\ell-2}{k-1}}{(\log(k+\ell-2))^c}.$$

Thomason [139] later published that for some constant c, for all $k \geq \ell \geq 1$

$$R(k+1,\ell+1) \leq e^{-(\ell/2k)\log k + c\sqrt{\log k}} \binom{k+\ell}{k}.$$

I have not worked out the details of how much better Thomason's bound is than Corollary 3.2.2, but according to *Math Reviews* MR968746 (90c:05152) it is often an improvement.

3.3 Constructive lower bounds

The following is perhaps one of the earliest known constructive bounds on the Ramsey numbers (see, *e.g.*, [4]). It could be considered a "folklore" bound.

Theorem 3.3.1. $R(k, \ell) > (k-1)(\ell-1)$.

Proof. The graph $G = (\ell-1)K_{k-1}$ contains no K_k, and the complement $\overline{G}$ contains no K_ℓ. □

Recall that given a matrix M, the *rank of* M, denoted $\mathrm{rank}(M)$, is the dimension of the row space of M. Also recall that for any matrices A and B such that AB exists, $\mathrm{rank}(A) \geq \mathrm{rank}(AB)$ (see, *e.g.*, [56, p. 159]).

Theorem 3.3.2 (Nagy, 1972 [101])**.** *For all* $t \geq 3$*,*

$$R(t+1, t+1) > \binom{t}{3}.$$

Proof. Let $t \geq 3$, $n = \binom{t}{3}$. Let K_n be the complete graph with $V(K_n) = [1, t]^3$. For $X, Y \in [1, t]^3$, colour the edge $\{X, Y\}$ red if $|X \cap Y| = 1$, and blue otherwise.

Let d be the largest number of vertices in a red clique in K_n, and let $A_1, \ldots, A_d$ be the vertices of this red clique. Let $\mathbf{v_1}, \ldots, \mathbf{v_d}$ be 0-1 incidence (or characteristic) vectors for $A_1, \ldots, A_d$ respectively (that is, each $\mathbf{v_i} = (v_{i,1}, \ldots, v_{i,n})$ is such that $v_{i,j} = 1$ if $j \in A_i$, and 0 otherwise). Then since each characteristic vector contains exactly three ones, for all $i \in [1, d]$, $\mathbf{v_i} \bullet \mathbf{v_i} = 3$. As well, for all $i, j \in [1, d]$, $i \neq j$, since each pair of sets A_i, A_j intersects at exactly one point, $\mathbf{v_i} \bullet \mathbf{v_j} = 1$. Let $M_{d \times t}$ be the matrix with the $\mathbf{v_i}$'s as rows (in order). Then $(MM^T)_{d \times d}$ is the square matrix with 3's down the diagonal and 1's everywhere else. Since MM^T is invertible,

$$d = \text{rank}(MM^T) \leq \text{rank}(M) \leq \min\{d, t\} \leq t.$$

Therefore d, the largest number of vertices in a red clique in K_n, is at most t, and so there is no red K_{t+1} in K_n.

Let c be the largest number of vertices in a blue clique in K_n, and let $B_1, \ldots, B_c$ be the vertices of such a clique. Let $\mathbf{u_1}, \ldots, \mathbf{u_c}$ be 0-1 incidence (or characteristic) vectors for $B_1, \ldots, B_c$ respectively. Then for all $i, j \in [1, c]$, $i \neq j$, $\mathbf{u_i} \bullet \mathbf{u_i} = 3$, and $\mathbf{u_i} \bullet \mathbf{u_j}$ is 0 or 2. Let $N_{c \times t}$ be the matrix with these vectors as rows. Then $(NN^T)_{c \times c}$ mod 2 is the $c \times c$ identity matrix. By arguments similar to those above,

$$c = \text{rank}(NN^T \bmod 2) \leq \text{rank}(NN^T) \leq \text{rank}(N) \leq \min\{c, t\} \leq t.$$

Then, since c is the largest number of vertices in a blue clique in K_n, there is no blue K_{t+1}. So since under the defined colouring of K_n there is no red K_{t+1} and no blue K_{t+1}, it follows that $R(t + 1, t + 1) > n = \binom{t}{3}$. □

Other explicit constructions have been found, including the following by Frankl and Wilson (proof is omitted):

Theorem 3.3.3 (Frankl and Wilson, 1981 [55]). *Let p be a prime. Let $n = \binom{p^3}{p^2-1}$, and let K_n be the complete graph with vertex set $[1, p^3]^{p^2-1}$. Given two sets $X, Y \in [1, p^3]^{p^2-1}$, colour $\{X, Y\}$ red if $|X \cap Y| \equiv -1 \mod p$, and blue otherwise. Then for $t = \binom{p^3}{p-1}$, K_n contains no monochromatic copy of K_t, and thus $R(t+1, t+1) > n$.*

I have not determined how much of an improvement Frankl and Wilson's construction is over Nagy's. For further known constructive results, see [4] or [123, p. 8].

3.4 Probabilistic lower bounds

3.4.1 The Erdős lower bound

The following are a few basic definitions in probability theory. For a more complete reference, see [8]. For present purposes, a *probability space* is a pair (Ω, P) where Ω is a finite set and $P : \Omega \to \{x \in \mathbb{R} : 0 \leq x \leq 1\}$ is a function such that $\sum_{v \in \Omega} P(v) = 1$. If for each $v \in \Omega$, $P(v) = \frac{1}{|\Omega|}$, then the probability space is *uniform*. An *event* is a subset of Ω, and define the *probability of an event A* as

$$P(A) = \sum_{v \in A} P(v).$$

Note that then $P(\Omega) = 1$.

It is likely that the most widely cited bound on the Ramsey numbers is the following theorem due to Erdős, which is also commonly used as a first example illustrating the power of the "probabilistic method".

Theorem 3.4.1 (Erdős, 1947 [34]). *For $n, k \in \mathbb{Z}^+$, if $\binom{n}{k}2^{1-\binom{k}{2}} < 1$ then*

$$R(k,k) \geq n+1.$$

Proof. Let $n, k \in \mathbb{Z}^+$ be such that $\binom{n}{k}2^{1-\binom{k}{2}} < 1$. Let Ω be the set of all graphs on $[1,n]$, and let (Ω, P) be the uniform probability space on Ω (that is, for every $G \in \Omega$, $P(G) = 2^{-\binom{n}{2}}$). For any $V \in [1,n]^k$, let A_V be the event that either V is a clique, or that V is an independent set. For each such V, the number of graphs that have V as a clique (which is $\frac{1}{2}|A_V|$) is $2^{\binom{n}{2}-\binom{k}{2}}$. Therefore $|A_V| = 2 \cdot 2^{\binom{n}{2}-\binom{k}{2}}$, and so for any $V \in [1,n]^k$,

$$P(A_V) = \sum_{G \in A_V} P(G) = \sum_{G \in A_V} 2^{-\binom{n}{2}} = 2 \cdot \left(2^{\binom{n}{2}-\binom{k}{2}}\right) 2^{-\binom{n}{2}} = 2^{1-\binom{k}{2}}.$$

Since there are $\binom{n}{k}$ choices for V, the probability that at least one $V \in [1,n]^k$ is such that A_V occurs is

$$P\left(\bigcup_{V \in [1,n]^k} A_V\right) \leq \sum_{V \in [1,n]^k} P(A_V) = \sum_{V \in [1,n]^k} 2^{1-\binom{k}{2}} = \binom{n}{k} 2^{1-\binom{k}{2}} < 1.$$

Thus, there is a positive probability that no event A_V occurs, and therefore there exists a graph on n vertices with no k-clique, and no independent set of k vertices. In other words, $R(k,k) \geq n+1$. $\square$

Corollary 3.4.2 (Erdős, 1947 [34]). *There exists a constant c such that for $k \geq 3$,*

$$R(k,k) > \frac{c}{e\sqrt{2}}\, k 2^{k/2}.$$

Proof. Let $k \geq 3$, and let

$$n = \left\lfloor \frac{1}{e\sqrt{2}}\, k2^{k/2} \left(\frac{\pi k}{2}\right)^{\frac{1}{2k}} e^{\frac{1}{k(12k+1)}} \right\rfloor .$$

Then,

$$\begin{aligned}
\binom{n}{k} 2^{1-\binom{k}{2}} &= \frac{n(n-1)\cdots(n-k+1)}{k!}\, 2^{1-\binom{k}{2}} \\
&< \frac{n^k}{k!}\, 2^{1-\binom{k}{2}} \\
&< n^k \frac{1}{\sqrt{2\pi k}} \left(\frac{e}{k}\right)^k e^{-\frac{1}{12k+1}}\, 2^{1-\binom{k}{2}} \qquad \text{(by Stirling's formula)} \\
&\leq \left(\frac{1}{e^k\, 2^{k/2}}\, k^k\, 2^{k^2/2} \left(\frac{\pi k}{2}\right)^{\frac{1}{2}} e^{\frac{1}{12k+1}}\right) \frac{1}{\sqrt{2\pi k}} \left(\frac{e}{k}\right)^k e^{-\frac{1}{12k+1}}\, 2^{1-\binom{k}{2}} \\
&= \left(\frac{2^{k^2/2}}{2^{k/2}}\right) \left(\frac{\pi k}{2}\right)^{\frac{1}{2}} \frac{1}{\sqrt{2\pi k}}\, 2^{1-k^2/2+k/2} \\
&= 1
\end{aligned}$$

Therefore by Theorem 3.4.1,

$$R(k,k) \geq n+1 > \frac{1}{e\sqrt{2}}\, k2^{\frac{k}{2}} \left(\frac{\pi k}{2}\right)^{\frac{1}{2k}} e^{\frac{1}{k(12k+1)}} > \frac{c}{e\sqrt{2}}\, k2^{\frac{k}{2}}. \qquad \square$$

3.4.2 The Lovász local lemma

The so-called *Lovász local lemma* (or simply *local lemma*, or LLL) was first presented and used in a 1975 paper by Erdős and Lovász [44] on extremal problems related to the class of k-chromatic r-uniform hypergraphs. The Lovász local lemma was used to produce a bound on the Ramsey numbers in the same year by Spencer [132]. The bound produced by the local lemma is only a constant factor better than the bound given in Corollary 3.4.2.

A few more definitions are needed first. A *random variable* in a probability space (Ω, P) is a function $X : \Omega \to \mathbb{R}$. If $X : \Omega \to \{0, 1\}$, then X is a *Bernoulli random variable*. The *expected value of a random variable* X is defined as

$$E(X) = \sum_{G \in \Omega} X(G)P(G).$$

Expectation is linear in the sense that for any two random variables X and Y, and any $a \in \mathbb{R}$, $E(aX + Y) = aE(X) + E(Y)$.

Given two events A and B in a probability space (Ω, P), the *conditional probability of A given B*, denoted $P(A \mid B)$ is defined as

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}.$$

Bayes' theorem states that for any events A,B, and C,

$$P(A \mid B \cap C) = \frac{P(A \cap B \mid C)}{P(B \mid C)},$$

and is used repeatedly in what follows.

Two events A and B in some probability space (Ω, P) are said to be *independent* iff $P(A \cap B) = P(A) \cdot P(B)$. For any $k \in \mathbb{Z}^+$, and events $B_1, \ldots, B_k$, an event A

is *mutually independent* of the set $\{B_1, \ldots, B_k\}$ iff A is independent of any boolean combination of the B_i's; in this case, the property used is that for any $I \subseteq [1, k]$,

$$P\left(A \cap \bigcap_{i \in I} B_i\right) = P(A) \cdot P\left(\bigcap_{i \in I} B_i\right).$$

Let $A_1, \ldots, A_n$ be events in some probability space (Ω, P), and let $G = (V, E)$ be a graph with $V(G) = [1, n]$. The graph G is said to be a *dependency graph for the events* $A_1, \ldots, A_n$ if for each $i \in [1, n]$, A_i is mutually independent of the set of events $\{A_j : \{i, j\} \notin E(G)\}$. By the definition of a dependency graph, its vertices must be integers, however the elements of the vertex set are not important in applications. Therefore, when convenient, a "dependency graph" can refer to any graph isomorphic to a dependency graph with integer vertices. Recall that in a graph with vertex i, $N(i)$ is the neighbourhood of i.

Theorem 3.4.3 (Lovász local lemma, Erdős and Lovász, 1975 [44])**.** *Let* (Ω, P) *be a finite probability space, and let* $A_1, \ldots, A_n$ *be events in this space. Let* G *be a dependency graph for* $A_1, \ldots, A_n$ *(with* $V(G) = [1, n]$*). If there exist* $x_1, \ldots, x_n \in \mathbb{R}$ *such that for all* $i \in [1, n]$, $0 \leq x_i < 1$, *and*

$$P(A_i) \leq x_i \prod_{j \in N(i)} (1 - x_j),$$

then

$$P\left(\bigcap_{i=1}^{n} \overline{A_i}\right) \geq \prod_{i=1}^{n} (1 - x_i) > 0.$$

Proof. Assume there exist $x_1, \ldots, x_n \in \mathbb{R}$ such that for all $i \in [1,n]$, $0 \leq x_i < 1$, and

$$P(A_i) \leq x_i \prod_{j \in N(i)} (1 - x_j).$$

In proving Theorem 3.4.3, the following claim is central.

Claim. For all $S \subsetneq [1,n]$, and for all $i \notin S$,

$$P\left(A_i \,\middle|\, \bigcap_{j \in S} \overline{A_j}\right) \leq x_i.$$

The proof of this claim is by strong induction on $|S|$. For any fixed $k \in [0, n-1]$, let $T(k)$ be the statement that for all $S \in [1,n]^k$, and for all $i \notin S$, $P\left(A_i \,\middle|\, \bigcap_{j \in S} \overline{A_j}\right) \leq x_i$. To prove the claim, it suffices to show that $T(0), T(1), \ldots, T(n-1)$ hold.

Base Case: For $k = 0$, $S = \emptyset$, and for any $i \in [1,n]$,

$$P\left(A_i \,\middle|\, \bigcap_{j \in S} \overline{A_j}\right) = P(A_i) \leq x_i \prod_{j \in N(i)} (1 - x_j) \leq x_i,$$

and therefore $T(0)$ holds.

Inductive Step: Let $k \in [0, n-2]$, and assume $T(0), \ldots, T(k)$ all hold. Let $S \in [1,n]^{k+1}$, and let $i \notin S$. Let $S_1 = N(i) \cap S$, $S_2 = S \setminus S_1$. Then

$$\begin{aligned} P\left(A_i \,\middle|\, \bigcap_{j \in S} \overline{A_j}\right) &= P\left(A_i \,\middle|\, \bigcap_{j \in S_1} \overline{A_j} \cap \bigcap_{j \in S_2} \overline{A_j}\right) \\ &= \frac{P\left(A_i \cap \left(\bigcap_{j \in S_1} \overline{A_j}\right) \mid \bigcap_{j \in S_2} \overline{A_j}\right)}{P\left(\bigcap_{j \in S_1} \overline{A_j} \mid \bigcap_{j \in S_2} \overline{A_j}\right)} \qquad \text{(by Bayes' theorem)} \end{aligned}$$

$$\begin{aligned}
&\le \frac{P\left(A_i \mid \bigcap_{j\in S_2} \overline{A_j}\right)}{P\left(\bigcap_{j\in S_1} \overline{A_j} \mid \bigcap_{j\in S_2} \overline{A_j}\right)} \\
&= \frac{P(A_i)}{P\left(\bigcap_{j\in S_1} \overline{A_j} \mid \bigcap_{j\in S_2} \overline{A_j}\right)} \\
&\le \frac{x_i \prod_{j\in N(i)}(1-x_j)}{P\left(\bigcap_{j\in S_1} \overline{A_j} \mid \bigcap_{j\in S_2} \overline{A_j}\right)}
\end{aligned}$$

If $S_1 = \emptyset$, then the denominator is 1, and since $x_i \prod_{j\in N(i)}(1-x_j) \le x_i$, $T(k)$ would hold, so assume $S_1 \ne \emptyset$. Without loss of generality, assume that (for some $r \ge 1$) $S_1 = [1, r]$, and $S_2 = [r+1, k+1]$. Then the denominator is

$$\begin{aligned}
P\left(\bigcap_{j\in S_1} \overline{A_j} \;\Big|\; \bigcap_{j\in S_2} \overline{A_j}\right) &= P\left(\bigcap_{i=1}^{r} \overline{A_i} \;\Big|\; \bigcap_{i=r+1}^{k+1} \overline{A_i}\right) \\
&= P\left(\overline{A_1} \;\Big|\; \bigcap_{i=2}^{k+1} \overline{A_i}\right) P\left(\bigcap_{i=2}^{r} \overline{A_i} \;\Big|\; \bigcap_{i=r+1}^{k+1} \overline{A_i}\right) \\
&= P\left(\overline{A_1} \;\Big|\; \bigcap_{i=2}^{k+1} \overline{A_i}\right) P\left(\overline{A_2} \;\Big|\; \bigcap_{i=3}^{k+1} \overline{A_i}\right) P\left(\bigcap_{i=3}^{r} \overline{A_i} \;\Big|\; \bigcap_{i=r+1}^{k+1} \overline{A_i}\right) \\
&\vdots \\
&= \prod_{j=1}^{r} P\left(\overline{A_j} \;\Big|\; \bigcap_{i=j+1}^{k+1} \overline{A_i}\right) \\
&= \prod_{j=1}^{r} \left[1 - P\left(A_j \;\Big|\; \bigcap_{i=j+1}^{k+1} \overline{A_i}\right)\right] \\
&\ge \prod_{j=1}^{r} (1-x_j)
\end{aligned}$$

$$\begin{aligned} &= \prod_{j\in S_1}(1-x_j) \\ &= \prod_{j\in N(i)\cap S}(1-x_j) \\ &\geq \prod_{j\in N(i)}(1-x_j). \end{aligned}$$

Therefore,

$$P\left(A_i \,\middle|\, \bigcap_{j\in S}\overline{A_j}\right) \leq \frac{x_i\prod_{j\in N(i)}(1-x_j)}{P(\bigcap_{j\in S_1}\overline{A_j} \mid \bigcap_{j\in S_2}\overline{A_j})} \leq \frac{x_i\prod_{j\in N(i)}(1-x_j)}{\prod_{j\in N(i)}(1-x_j)} = x_i,$$

and so $T(k+1)$ holds. Therefore, by mathematical induction, $T(0),\ldots,T(n-1)$ all hold, and so the claim holds.

To finish proving the theorem, all that is needed are the two laws:

$$P(A\cap B) = P(A)\cdot P(B \mid A) \tag{3.1}$$

$$P(A\cap B \mid C) = P(A \mid C)\cdot P(B \mid A\cap C). \tag{3.2}$$

Then

$$\begin{aligned} P\left(\bigcap_{i=1}^{n}\overline{A_i}\right) &= P\left(\overline{A_1}\right)\cdot P\left(\bigcap_{i=2}^{n}\overline{A_i} \,\middle|\, \overline{A_1}\right) && \text{(by equation (3.1))} \\ &= P\left(\overline{A_1}\right)\cdot P\left(\overline{A_2} \mid \overline{A_1}\right)\cdot P\left(\bigcap_{i=3}^{n}\overline{A_i} \,\middle|\, \overline{A_2}\cap\overline{A_1}\right) && \text{(by equation (3.2))} \\ &\vdots \\ &= \prod_{i=1}^{n} P\left(\overline{A_i} \,\middle|\, \bigcap_{j=1}^{i-1}\overline{A_j}\right) \\ &= \prod_{i=1}^{n}\left[1-P\left(A_i \,\middle|\, \bigcap_{j=1}^{i-1}\overline{A_j}\right)\right] \end{aligned}$$

$$\geq \prod_{i=1}^{n}(1 - x_i) \qquad \text{(by the claim with } S = [1, i-1]\text{).} \qquad \square$$

Corollary 3.4.4 (Lovász local lemma, Symmetric version [132])**.** *Let $A_1, \ldots, A_n$ be events in the probability space (Ω, P), and assume that there exists p such that $0 \leq p \leq 1$ and for all $i \in [1, n]$, $P(A_i) \leq p$. Let G be a dependency graph for $A_1, \ldots, A_n$, and let $d \in \mathbb{Z}^+$ be such that the maximum degree of G is at most d. Then if $ep(d+1) \leq 1$, then $P\left(\bigcap_{i=1}^{n} \overline{A_i}\right) > 0$.*

Proof. For all $i \in [1, n]$, let $x_i = \frac{1}{d+1}$, and assume that $ep(d+1) \leq 1$. Then

$$\begin{aligned}
x_i \prod_{j \in N(i)} (1 - x_j) &= \frac{1}{d+1}\left(1 - \frac{1}{d+1}\right)^{\deg(i)} \\
&\geq \frac{1}{d+1}\left(1 - \frac{1}{d+1}\right)^{d} \\
&= \frac{1}{d+1}\left(\frac{d}{d+1}\right)^{d} \\
&= \frac{1}{d+1}\left(\frac{d+1}{d}\right)^{-d} \\
&= \frac{1}{d+1}\left(1 + \frac{1}{d}\right)^{-d} \\
&> \frac{1}{d+1}(e^{1/d})^{-d} \qquad \text{(since if } x > 0,\ e^x > 1 + x\text{)} \\
&= \frac{1}{d+1}\frac{1}{e} \\
&\geq p \qquad \text{(since } ep(d+1) \leq 1\text{)} \\
&\geq P(A_i).
\end{aligned}$$

Therefore LLL applies, and thus, $P\left(\bigcap_{i=1}^{n} \overline{A_i}\right) > 0$. $\square$

Note: the requirement of Corollary 3.4.4 for $ep(d+1) \leq 1$ has been on occasion weakened to $4pd \leq 1$ (see, *e.g.*, [134, p. 57]).

3.4.3 Lower bound on $R(k,k)$ using the Lovász local lemma

Theorem 3.4.5. *For any $n, k \geq 2$, if*

$$e\left(2^{1-\binom{k}{2}}\right)\left(\binom{k}{2}\binom{n}{k-2}\right) \leq 1,$$

then $R(k,k) \geq n+1$.

Proof. Let $n, k \geq 2$, and let Δ be a random 2-colouring of the edges of K_n where the probability that a given edge is red is $\frac{1}{2}$ (that is, Ω is the set of all edge 2-colourings of K_n, the probability space (Ω, P) is the uniform probability space on Ω, and Δ is an element of Ω). For each set S of k vertices in K_n, let A_S be the event that $\binom{S}{K_2}$ is monochromatic under Δ (that is, A_S is the set of edge 2-colourings that make $\binom{S}{K_2}$ monochromatic). Then

$$P(A_S) = 2 \cdot \left(\frac{1}{2}\right)^{\binom{k}{2}} = 2^{1-\binom{k}{2}}.$$

If S and T are two sets of k vertices and $|S \cap T| \geq 2$, then A_S and A_T are not independent. Let G be a dependency graph for the A_S's. Then each vertex in G is connected to at most

$$\binom{k}{2}\binom{n-2}{k-2} - 1 < \binom{k}{2}\binom{n}{k-2} - 1$$

other vertices. Therefore the maximum degree of G is at most $\binom{k}{2}\binom{n}{k-2} - 1$.

Thus by the symmetric Lovász local lemma (Corollary 3.4.4), if

$$e\left(2^{1-\binom{k}{2}}\right)\left(\binom{k}{2}\binom{n}{k-2}\right) \leq 1,$$

then $P\left(\bigcap_S \overline{A_S}\right) > 0$, in which case there exists a colouring of the edges of K_n containing no monochromatic k-set, *i.e.*, $R(k,k) \geq n+1$. □

If two real-valued functions on the integers f and g satisfy

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0,$$

then write $f(x) = o(g(x))$, and say that f is *little-oh* of g. A function that is $o(1)$ is then a function that approaches 0. For convenience, the term $o(1)$ is used to represent an arbitrary function that is $o(1)$. This is a type of "Landau notation".

Corollary 3.4.6. *For all $k \in \mathbb{Z}^+$,*

$$R(k,k) > \frac{\sqrt{2}}{e}\, k2^{k/2}(1+o(1)).$$

Proof. Let $k \in \mathbb{Z}^+$, and assume

$$e\left(2^{1-\binom{k}{2}}\right)\left(\binom{k}{2}\binom{n}{k-2}\right) \leq 1.$$

Then solving for $\binom{n}{k-2}$,

$$\binom{n}{k-2} \leq \frac{2^{\binom{k}{2}}}{2e\binom{k}{2}}.$$

Therefore since $\binom{n}{k-2} \leq \left(\frac{ne}{k-2}\right)^{k-2}$, it suffices to find n such that

$$\left(\frac{ne}{k-2}\right)^{k-2} \leq \frac{2^{\binom{k}{2}}}{ek(k-1)}$$

Then by solving for n and simplifying,

$$\begin{aligned}
\left(\frac{ne}{k-2}\right)^{k-2} &\leq \frac{2^{\binom{k}{2}}}{ek(k-1)} \\
\iff n &\leq \frac{2^{\frac{k(k-1)}{2(k-2)}}}{(ek(k-1))^{\frac{1}{k-2}}}\frac{k-2}{e} \\
\iff n &\leq k2^{k/2}\left(\frac{2^{\frac{k}{2(k-2)}}(1-\frac{2}{k})}{e(ek(k-1))^{\frac{1}{k-2}}}\right) \\
\iff n &\leq \frac{\sqrt{2}}{e}k2^{k/2}\left(\frac{2^{\frac{1}{k-2}}(1-\frac{2}{k})}{(ek(k-1))^{\frac{1}{k-2}}}\right) \\
\implies n &\leq \frac{\sqrt{2}}{e}k2^{k/2}\,(1+o(1)) \qquad \left(\text{since } \lim_{k\to\infty}\frac{2^{\frac{1}{k-2}}(1-\frac{2}{k})}{(ek(k-1))^{\frac{1}{k-2}}} = 1\right). \qquad \square
\end{aligned}$$

Putting the upper and lower bounds together, there exists some constants c_1, c_2 such that the Ramsey numbers are bounded by

$$c_1k\ 2^{\frac{k}{2}} \leq R(k,k) \leq \frac{c_2}{\sqrt{k}}\ 2^{2k}. \tag{3.3}$$

3.4.4 Bounding $R(3,t)$

By probabilistic methods, Erdős [36] gave a lower bound for $R(3,t)$ of the form $c\frac{t^2}{(\log t)^2}$. Spencer used the local lemma to get the same result with a simpler proof.

Lemma 3.4.7 (Spencer, 1977 [133]). *If $n, t \in \mathbb{Z}^+$, $p \in (0,1)$, and $x, y \in [0,1)$ are such that*

$$p^3 \le x(1-x)^{3n}(1-y)^{\binom{n}{t}}, \text{ and} \tag{3.4}$$

$$(1-p)^{\binom{t}{2}} \le y(1-x)^{\frac{t^2 n}{2}}(1-y)^{\binom{n}{t}} \tag{3.5}$$

hold, then $R(3,t) \ge n$.

Proof. Let n, t, p, x, y be as above. Let (Ω, P) be the probability space of all edge 2-colourings of K_n, where for any $\Delta : E(K_n) \to \{\text{red}, \text{blue}\}$, if $r = |\Delta^{-1}(\text{red})|$, then

$$P(\Delta) = p^r(1-p)^{\binom{n}{2}-r}.$$

That is, each edge of K_n is independently coloured red with probability p. Let $\mathcal{X} = [V(K_n)]^3$, $\mathcal{Y} = [V(K_n)]^t$. For each $S \in \mathcal{X}$, let A_S be the event that all the edges in S are coloured red, and for each $T \in \mathcal{Y}$, let B_T be the event that all the edges in T are coloured blue. Note that for any $S \in \mathcal{X}, T \in \mathcal{Y}$, $P(A_S) = p^3$, and $P(B_T) = (1-p)^{\binom{t}{2}}$.

Let G be the graph with $V(G) = \mathcal{X} \cup \mathcal{Y}$ such that for any $Z_1, Z_2 \in V(G)$, $\{Z_1, Z_2\} \in E(G)$ if and only if $|Z_1 \cap Z_2| \ge 2$. Then G is a dependency graph for the events

$$\{A_S : S \in \mathcal{X}\} \cup \{B_T : T \in \mathcal{Y}\}.$$

Let $\alpha = |\mathcal{X}|$, $\beta = |\mathcal{Y}|$, and enumerate $\{A_S : S \in \mathcal{X}\} = \{A_1, \ldots, A_\alpha\}$ and $\{B_T : T \in \mathcal{Y}\} = \{B_1, \ldots, B_\beta\}$. By the Lovász local lemma (Theorem 3.4.3), if there exist

$x_1, \ldots, x_\alpha, y_1, \ldots, y_\beta \in [0,1)$ such that for all $i \in [1, \alpha]$,

$$P(A_i) \leq x_i \prod_{A_j \in N_G(A_i) \cap \mathcal{X}} (1 - x_j) \prod_{B_j \in N_G(A_i) \cap \mathcal{Y}} (1 - y_j),$$

and for all $i \in [1, \beta]$,

$$P(B_i) \leq y_i \prod_{A_j \in N_G(B_i) \cap \mathcal{X}} (1 - x_j) \prod_{B_j \in N_G(B_i) \cap \mathcal{Y}} (1 - y_j),$$

then

$$P\left(\bigcap_{i=1}^{\alpha} \overline{A_i} \cap \bigcap_{j=1}^{\beta} \overline{B_j}\right) > 0,$$

implying that there exists a colouring $\Delta \in \Omega$ exhibiting no monochromatic red K_3 and no monochromatic blue K_t, and therefore $R(3,t) > n$.

Note that for every $A_i \in \mathcal{X}$ and every $B_j \in \mathcal{Y}$,

$$\begin{aligned}
|N_G(A_i) \cap \mathcal{X}| &= 3(n-3) < 3n, && (3.6)\\
|N_G(B_j) \cap \mathcal{X}| &= \binom{t}{2}(n-t) + \binom{t}{3} < \binom{t}{2} n < \frac{t^2 n}{2}, && (3.7)\\
|N_G(A_i) \cap \mathcal{Y}| &\leq |\mathcal{Y}| = \binom{n}{t}, && (3.8)\\
|N_G(B_j) \cap \mathcal{Y}| &\leq |\mathcal{Y}| = \binom{n}{t}. && (3.9)
\end{aligned}$$

Define $x_1 = x,\ x_2 = x, \ldots,\ x_\alpha = x$, and define $y_1 = y,\ y_2 = y, \ldots,\ y_\beta = y$ (where x and y were defined in the statement of the theorem). Then, for any $i \in [1, \alpha]$,

$$\begin{aligned}
P(A_i) &= p^3, \\
&\leq x(1-x)^{3n}(1-y)^{\binom{n}{t}}, && \text{(by (3.4))}
\end{aligned}$$

$$\begin{aligned}
&< x(1-x)^{|N_G(A_i)\cap\mathcal{X}|}(1-y)^{|N_G(A_i)\cap\mathcal{Y}|}, && \text{(by (3.6) and (3.8))}\\
&= x_i \prod_{A_j\in N_G(A_i)\cap\mathcal{X}}(1-x_j)\prod_{B_j\in N_G(A_i)\cap\mathcal{Y}}(1-y_j).
\end{aligned}$$

Similarly, for any $i\in[1,\beta]$,

$$\begin{aligned}
P(B_i) &= (1-p)^{\binom{t}{2}}, &&\\
&\le y(1-x)^{\frac{t^2n}{2}}(1-y)^{\binom{n}{t}}, && \text{(by (3.5))}\\
&< y(1-x)^{|N_G(B_i)\cap\mathcal{X}|}(1-y)^{|N_G(B_i)\cap\mathcal{Y}|}, && \text{(by (3.7) and (3.9))}\\
&= y_i \prod_{A_j\in N_G(B_i)\cap\mathcal{X}}(1-x_j)\prod_{B_j\in N_G(B_i)\cap\mathcal{Y}}(1-y_j).
\end{aligned}$$

Therefore, the x_i's and the y_i's work, and so by the Lovász local lemma,

$$P\left(\bigcap_{i=1}^{\alpha}\overline{A_i}\cap\bigcap_{j=1}^{\beta}\overline{B_j}\right)>0. \qquad \square$$

Corollary 3.4.8. *There exists a constant c such that for any $t\ge 3$,*

$$R(3,t)>c\frac{t^2}{(\log t)^2}.$$

To actually show that $R(3,t)>c\frac{t^2}{(\log t)^2}$, it suffices to show that for any $t\in\mathbb{Z}^+$ and for $n=c\frac{t^2}{(\log t)^2}$, there exist $p\in(0,1)$ and $x,y\in[0,1)$ such that the inequalities (3.4) and (3.5) hold. Spencer says that "elementary analysis (and a free weekend!)" [134, p. 63] can be used to show that $n=c\frac{t^2}{(\log t)^2}$ works, and is optimal. I have personally not worked through these details (see [8, pp. 286–289] for the details).

Using the same argument as in the proof of Lemma 3.4.7, Spencer also proved the following generalization.

Theorem 3.4.9 (Spencer, 1977 [133]). *For any $m \in \mathbb{Z}^+$, there exists a constant $c = c(m)$ such that for all $t \geq m$,*

$$R(m,t) > c\left(\frac{t}{\log t}\right)^{\left(\binom{m}{2}-1\right)/(m-2)}.$$

A notable consequence of Theorem 3.4.9 is the bound $R(4,t) > c\left(\frac{t}{\log t}\right)^{5/2}$, which, according to Alon and Spencer [3, p. 68], is still the best known lower bound for general $R(4,t)$.

In terms of upper bounds, in 1968 Graver and Yackel [66] showed that $R(3,t) \leq c \log\log t \frac{t^2}{\log t}$. Graver and Yackel's bound was later improved by Ajtai, Komlós, and Szemerédi in 1980 [2], who proved that $R(3,t) \leq c\frac{t^2}{\log t}$.

Spencer's bound for $R(3,t)$ was improved by Kim in 1995 [82] to show that $R(3,t) \geq c\frac{t^2}{\log t}$. The result of Ajtai *et al.*, together with Kim's result, shows that there are constants c_1 and c_2 such that

$$c_1\frac{t^2}{\log t} \leq R(3,t) \leq c_2\frac{t^2}{\log t}.$$

In other words, the proper order of magnitude was the bound by Ajtai, Komlós, and Szemerédi.

3.5 Bounds for $R_k(m;2)$ when $k>2$

When colouring general k-tuples, not much is known other than the following bound:

Theorem 3.5.1 (Erdős, 1947 [34] and Erdős, and Rado, 1952 [45]). *For any $k \in \mathbb{Z}^+$, there exist constants c_1, c_2 depending only on k such that for all m,*

$$2^{c_1 m^{k-1}/k!} < R_k(m;2) < \underbrace{2^{2^{\cdot^{\cdot^{\cdot^{2^{c_2 m}}}}}}}_{k-1}$$

Corollary 3.5.2. *There exists constants c_1, c_2 such that*

$$2^{c_1 m^2} < R_3(m;2) < 2^{2^{c_2 m}}.$$

For discussion on the origins of Theorem 3.5.1, see [24, p. 30], [38, p. 18], and [43, p. 140]. For other known bounds on the Ramsey numbers $R(a,b)$ and $R_k(m;r)$, see, *e.g.*, [123] or [142].

Chapter 4

Weak graph Ramsey numbers

The values $R(G; K_2; 2)$ are known as the "diagonal weak graph Ramsey numbers", and in this chapter an "off-diagonal" analog is also defined. Determining both the diagonal and off-diagonal values has become an area of considerable study in graph Ramsey theory since the late 1960s. Much of the work in this field is largely due to Burr, Chvátal, Erdős, Faudree, Rousseau, and Schelp. This chapter presents some theorems representative of determining both the on- and off-diagonal weak graph Ramsey numbers.

4.1 Preliminaries

The reader is referred to Appendix A for standard graph theory definitions. For any graphs G and H, denote the set of weak subgraphs of G isomorphic to H as $\binom{G}{H}$, that is,

$$\binom{G}{H} = \{H' \subseteq G : H' \cong H\}.$$

As discussed in the introduction, the standard is often seen as using $\binom{G}{H}$ to denote instead the set of induced subgraphs of G isomorphic to H (see, *e.g.*, [102]).

For any $r \in \mathbb{Z}^+$, and any graphs F, G and H, write

$$F \longrightarrow (G)_r^H$$

iff for any r-colouring $\Delta : \binom{F}{H} \to [1, r]$, there exists $G' \in \binom{F}{G}$ such that $\binom{G'}{H}$ is monochromatic under Δ. Ramsey's theorem implies that for any $k, m, r \in \mathbb{Z}^+$, there exists $n = R_k(m; r)$ such that $K_n \longrightarrow (K_m)_r^{K_k}$. Generalizing the Ramsey number $R_k(m; r)$, for graphs G, H, and $r \in \mathbb{Z}^+$, let $n = R(G; H; r)$ be the least integer (if any exists) such that

$$K_n \longrightarrow (G)_r^H.$$

The numbers $R(G; H; r)$, when they exist, are *graph Ramsey numbers*. The following observation echoes a statement made in the introduction:

Observation 4.1.1. *For any $k, m, r \in \mathbb{Z}^+$, $R(K_m; K_k; r) = R_k(m; r)$, and for any graph G on m vertices, $R(G; K_k; r) \leq R(K_m; K_k; r)$.*

This chapter deals strictly in the case when $k = 2$ (edge colourings). In the same way the Ramsey numbers were generalized to the "off-diagonal" version $R_k(a, b)$ in Chapter 3, graph Ramsey numbers can be generalized to off-diagonal versions as well. Recall from the introduction that for any graphs G_1 and G_2,

$$F \longrightarrow (G_1, G_2)_2^{K_2}$$

means that for any 2-colouring $\Delta : E(F) \to \{\text{red}, \text{blue}\}$, there exists either $G_1' \in \binom{F}{G_1}$ such that $E(G_1')$ is monochromatic red, or $G_2' \in \binom{F}{G_2}$ such that $E(G_2')$ is monochromatic blue. Let $R(G_1, G_2)$ be the least integer n such that

$$K_n \longrightarrow (G_1, G_2)_2^{K_2}.$$

Note that $R(G_1, G_2) \leq R(G_1 \dot{\cup} G_2; K_2; 2)$, and $R(G; K_2; 2) = R(G, G)$.

4.2 Graph Ramsey numbers for paths

One of the earliest theorems regarding the numbers $R(G_1, G_2)$ determined the exact value whenever G_1 and G_2 are both paths. Recall that the number of edges in a path P is called the *length* of P, and for any $k \in \mathbb{Z}^+$, P_k denotes a path of length k (which has $k + 1$ vertices).

Theorem 4.2.1 (Gerencsér and Gyárfás, 1967 [57]). *For all $k, \ell \in \mathbb{Z}^+$, $k \leq \ell$,*

$$R(P_k, P_\ell) = \left\lfloor \frac{k+1}{2} \right\rfloor + \ell,$$

and therefore $R(P_k, P_k) = \left\lceil \frac{3}{2}k \right\rceil$.

Proof. To see that $R(P_k, P_\ell) > \left\lfloor \frac{k+1}{2} \right\rfloor + \ell - 1$, let $k, \ell \in \mathbb{Z}^+$, $k \leq \ell$, $n = \left\lfloor \frac{k+1}{2} \right\rfloor + \ell - 1$. Let A be any set of ℓ vertices in K_n, and let $B = V(K_n) \setminus A$. Let Δ be the the 2-colouring of the edges of K_n shown in Figure 4.1. Under Δ, the largest blue path

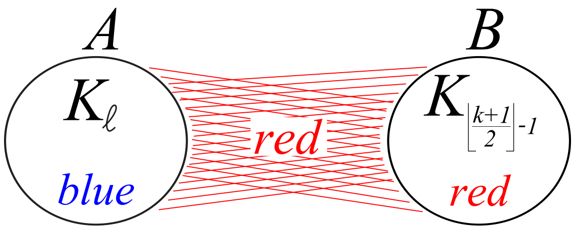


Figure 4.1: 2-colouring of K_n with no red P_k and no blue P_ℓ

has ℓ vertices.

Claim 1. Every red path of maximum length in K_n starts and ends in A, and goes back and forth between A and B.

Proof of Claim 1. If $k = 1$, $B = \emptyset$, so assume $k > 1$. Let $P = v_1, \ldots, v_N$ be any red path of maximum length, and let $P' = P \setminus \{v_N\}$. Note that by the maximality of P,

$$\text{for every } i \in [1, N-1], \text{ if } v_i \in A, \text{ then } v_{i+1} \in B. \tag{4.1}$$

Therefore,

$$\begin{aligned}
|P \cap A| &\leq |P' \cap A| + 1 \\
&\leq |P \cap B| + 1 \qquad \text{(by (4.1))} \\
&\leq |B| + 1 \\
&= \left\lfloor \frac{k+1}{2} \right\rfloor \\
&< k \qquad\qquad \text{(since } k > 1\text{)} \\
&\leq \ell.
\end{aligned}$$

Therefore $|P \cap A| < \ell$, and so there exists $x \in A \setminus P$.

If for any particular $i \in [1, N-1]$, the vertices v_i and v_{i+1} are both in B, then

$$v_1, v_2, \ldots, v_i, x, v_{i+1}, \ldots, v_N$$

would be a longer red path. Similarly, if $v_1 \in B$, or $v_n \in B$, then P could be extended at the beginning or end respectively using x. This proves Claim 1.

The maximum length of a path starting and ending in A, and going back and forth between A and B is

$$|B| + 1 = \left\lfloor \frac{k+1}{2} \right\rfloor < k.$$

This proves that $R(P_k, P_\ell) > \left\lfloor \frac{k+1}{2} \right\rfloor + \ell - 1$.

Gerencsér and Gyárfás' proof that $R(P_k, P_\ell) \leq \left\lfloor \frac{k+1}{2} \right\rfloor + \ell$ is by induction on ℓ.

For any fixed $k, \ell \in \mathbb{Z}^+$, $k \leq \ell$, let $S(k,\ell)$ denote the statement that $R(P_k, P_\ell) \leq \left\lfloor \frac{k+1}{2} \right\rfloor + \ell$.

Base Case. The statement $S(1,1)$ says that $R(P_1, P_1) \leq \left\lfloor \frac{1+1}{2} \right\rfloor + 1 = 2$, which holds.

Inductive Step. Fix $b \in \mathbb{Z}^+$, $b > 1$, and assume that for every $a \leq b$, $S(a,b)$ holds. It remains to show that for every $a \leq b+1$, $S(a, b+1)$ holds.

Fix $a \leq b+1$, $n = \left\lfloor \frac{a+1}{2} \right\rfloor + (b+1)$, and let $\Delta : E(K_n) \to \{\text{red}, \text{blue}\}$. Assume that under Δ there is no blue P_{b+1}.

There are two main cases: when $a \leq b$, and when $a = b+1$. If $a \leq b$, then since $S(a,b)$ holds,

$$R(P_a, P_b) \leq \left\lfloor \frac{a+1}{2} \right\rfloor + b = n - 1,$$

and therefore under Δ there is either a red P_a (and the proof is done) or there is a blue P_b. Let $P = v_1, \ldots, v_{b+1}$ denote this blue copy of P_b. Enumerate $V(K_n) \setminus P = X = \{x_1, \ldots, x_{\lfloor \frac{a+1}{2} \rfloor}\}$. Since there is no blue P_{b+1}, the path P must be maximal. Therefore, the following two properties hold:

(1) $\forall i \in [1,b], \forall j \in [1, \left\lfloor \frac{a+1}{2} \right\rfloor]$, either $\{x_j, v_i\}$ or $\{x_j, v_{i+1}\}$ is red, and

(2) $\forall j \in [1, \left\lfloor \frac{a+1}{2} \right\rfloor], \{x_j, v_1\}$ and $\{x_j, v_{b+1}\}$ are both red.

Let S be a maximal red path, say on s vertices, with end points A and B in X, not containing v_1 or v_{b+1}, and such that each edge of S connects a vertex in P with a vertex in X. If $X \subseteq S$, then S has length

$$2(|X|-1) = 2\left\lfloor \frac{a+1}{2} \right\rfloor - 2 \geq 2\left(\frac{a}{2}\right) - 2 = a-2,$$

and, by property (2), adding the edges $\{v_1, A\}$ and $\{v_{b+1}, B\}$ would form a red P_a, completing the proof.

Let T be a maximal red path, say on t vertices, disjoint from S, with end points C and D in X, not containing v_1 or v_{b+1}, and such that each edge of T connects a vertex in P with a vertex in X (see Figure 4.2). Note that the path T may be as small as a single vertex (*i.e.* a path of length 0).

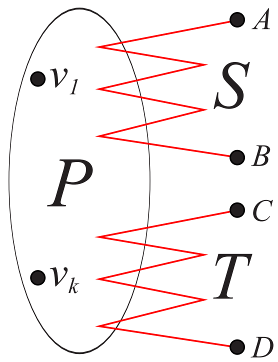


Figure 4.2: The paths S and T

Claim 2. Every vertex in X is either on S or T.

Proof of Claim 2. Assume there exists $u \in X$ not on S or T. Then the number of vertices in P on either S or T is

$$\frac{s-1}{2} + \frac{t-1}{2} \quad = \quad \frac{s+1}{2} + \frac{t+1}{2} - 2$$

$$\begin{aligned}
&\leq (|X|-1)-2 && \text{(by assumption)}\\
&= \left(\left\lfloor\frac{a+1}{2}\right\rfloor-1\right)-2\\
&\leq \left\lfloor\frac{b}{2}\right\rfloor-2 && \text{(since } a\leq b+1)\\
&< \left\lfloor\frac{b-1}{2}\right\rfloor\\
&\leq \frac{b-1}{2}
\end{aligned}$$

Therefore there exists $i_0 \in [2, b-1]$ such that v_{i_0} and v_{i_0+1} are not on S or T. By property (1), at least one of v_{i_0} and v_{i_0+1} is connected to u by a red edge (without loss of generality, say v_{i_0+1}). Then by the maximality of S and T, both $\{v_{i_0+1}, A\}$ and $\{v_{i_0+1}, C\}$ must be blue. Then, however, again by property (1), $\{v_{i_0}, A\}$ and $\{v_{i_0}, C\}$ must both be red, a contradiction to the maximality of S. This proves Claim 2.

Now, the circuit $Z = v_1, S, v_k, T, v_1$ is red, and has length

$$\begin{aligned}
s+t+2 &= 2\left(\frac{s+1}{2}+\frac{t+1}{2}\right)\\
&= 2|X| && \text{(by Claim 2)}\\
&= 2\left\lfloor\frac{a+1}{2}\right\rfloor.
\end{aligned}$$

If a is odd, then Z contains a red P_a, and the proof is done. Otherwise, Z has $2\left\lfloor\frac{a+1}{2}\right\rfloor = a$ vertices, exactly half of which are in P. Since v_1 and v_{b+1} are both in Z, and since $\frac{a}{2} \leq \frac{b+1}{2}$, there exists $j_0 \in [1, b]$ such that neither v_j nor v_{j+1} is in Z.

By property (1), one of v_j and v_{j+1} connects to the circuit by a red edge, creating a red P_a. This proves that for every $a \leq b$, $S(a, b+1)$ holds.

Finally, when $a = b+1$, since $S(b,b)$ holds, and thus

$$R(P_b, P_b) \leq \left\lfloor \frac{b+1}{2} \right\rfloor + b = \left\lfloor \frac{a}{2} \right\rfloor + b < \left\lfloor \frac{a+1}{2} \right\rfloor + b + 1 = n,$$

there is either a red P_b or a blue P_b. Without loss of generality, one can assume that there is a blue P_b and no blue P_{b+1}. From this point, the same proof as above works, producing a red $P_a = P_{b+1}$.

Therefore, by induction, for every $k, \ell \in \mathbb{Z}^+$, $k \leq \ell$, $S(k, \ell)$ holds, proving the theorem. $\square$

4.3 Small values

Many more values have been found for weak graph Ramsey numbers $R(G, H)$ than for the traditional Ramsey numbers $R(a, b)$. Several tables of small diagonal and off-diagonal graph Ramsey numbers were produced in the 1970s and 1980s (see, *e.g.*, [15, 26, 29, 52, 74]). Chvátal and Harary [27] produced a table of the graph Ramsey numbers values for all graphs with 4 or fewer vertices (see Table 4.1).

In 1989, Hendry [75] tabulated the off-diagonal graph Ramsey numbers for all but

$R(G,H)$	K_2	P_2	$2K_2$	K_3	P_3	$K_{1,3}$	C_4	$K_{1,3}+e$	K_4-e	K_4
K_2	2	3	4	3	4	4	4	4	4	4
P_2		3	4	5	4	5	4	5	5	7
$2K_2$			5	5	5	5	5	5	5	6
K_3				6	7	7	7	7	7	9
P_3					5	5	5	7	7	10
$K_{1,3}$						6	6	7	7	10
C_4							6	7	7	10
$K_{1,3}+e$								7	7	10
$K_4 \setminus e$									10	11
K_4										18

Table 4.1: $R(G,H)$ for graphs with at most four vertices, and no isolated vertices.

seven pairs of graphs with at most 5 vertices not containing isolated vertices (see Table 4.2).

Since Hendry produced Table 4.2, four of the missing seven have been found (displayed in bold in the table). First, Hendry stated (without proof or reference) that

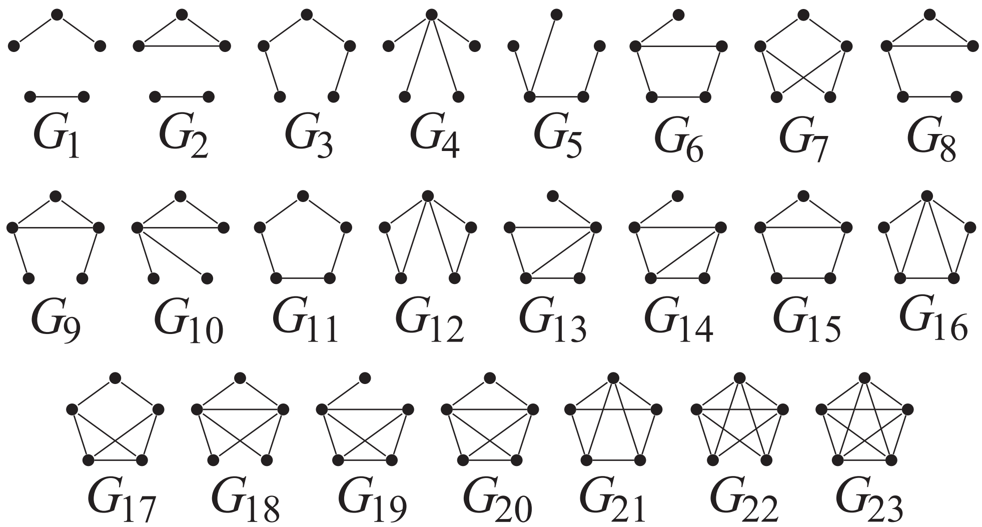


Figure 4.3: Graphs with exactly five vertices and no isolated vertices

$R(G_i, G_j)$	G_1	G_2	G_3	G_4	G_5	G_6	G_7	G_8	G_9	G_{10}	G_{11}	G_{12}	G_{13}	G_{14}	G_{15}	G_{16}	G_{17}	G_{18}	G_{19}	G_{20}	G_{21}	G_{22}	G_{23}
G_1	6	7	6	6	6	6	6	7	6	6	7	7	6	7	7	7	7	7	7	7	7	7	9
G_2		7	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	11	11	14
G_3			6	7	6	6	6	9	9	9	9	9	9	9	9	9	9	9	13	13	9	13	17
G_4				7	7	7	9	9	9	9	9	9	9	9	9	9	9	9	13	13	9	13	17
G_5					6	6	7	9	9	9	9	9	9	9	9	9	9	9	13	13	9	13	17
G_6						6	8	9	9	9	9	9	9	9	9	9	9	9	13	13	9	13	17
G_7							10	9	9	9	9	10	10	10	9	10	10	13	14	14	13	17	20
G_8								9	9	9	9	9	9	9	9	9	9	9	13	13	11	13	17
G_9									9	9	9	9	9	9	9	9	9	9	13	13	11	13	17
G_{10}										9	9	9	9	9	9	9	9	9	13	13	11	13	17
G_{11}											9	9	9	9	9	9	9	10	13	13	9	13	17
G_{12}												9	9	9	9	9	10	10	13	13	11	13	17
G_{13}													10	10	9	10	10	11	13	13	11	13	17
G_{14}														10	9	10	10	11	13	13	11	13	17
G_{15}															9	9	9	10	13	13	11	13	17
G_{16}																10	10	13	13	13	13	17	19
G_{17}																	10	13	14	14	13	17	20
G_{18}																		14	14	14	15	17	**20**
G_{19}																			18	18	17	19	**25**
G_{20}																				18	17	19	25-28
G_{21}																					15	**17**	**27**
G_{22}																						22	30-34
G_{23}																							43-49

Table 4.2: $R(G_i, G_j)$ for graphs with exactly five vertices and no isolated vertices

$R(G_{19}, G_{23}) = R(K_4, K_5)$, and so when $R(4, 5)$ was found to be 25 (by [80] and [96]), $R(G_{19}, G_{23})$ was also determined. The others found since are:

$$\begin{aligned} R(G_{18}, G_{23}) &= R(K_5 \setminus E(K_3), K_5) = 20 \text{ [5, 75]}, \\ R(G_{21}, G_{22}) &= R(K_5 \setminus E(2K_2), K_5 \setminus e) = 17 \text{ [75, 144], and} \\ R(G_{21}, G_{23}) &= R(K_5 \setminus E(2K_2), K_5) = 27 \text{ [75, 124]}. \end{aligned}$$

The remaining unknown values are

$$\begin{aligned} R(G_{20}, G_{23}) &= R(K_5 \setminus E(P_2), K_5), \\ R(G_{22}, G_{23}) &= R(K_5 \setminus e, K_5), \text{ and} \\ R(G_{23}, G_{23}) &= R(K_5, K_5) = R(5, 5). \end{aligned}$$

4.4 Other weak graph Ramsey numbers

This section includes selected bounds for other weak graph Ramsey numbers. For any graph G, let $c(G)$ denote the cardinality of the largest component of G.

Theorem 4.4.1 (Chvátal and Harary, 1972 [27]). *For any graphs G_1 and G_2,*

$$R(G_1, G_2) \geq (\chi(G_1) - 1)(c(G_2) - 1) + 1.$$

Proof. Let $n = (\chi(G_1) - 1)(c(G_2) - 1)$, and consider the graph K_n viewed as $\chi(G_1) - 1$ copies of $K_{c(G_2)-1}$ with edges interconnecting every possible pair of vertices (see Figure 4.4). Colour all the edges within each $K_{c(G_2)-1}$ blue, and colour each

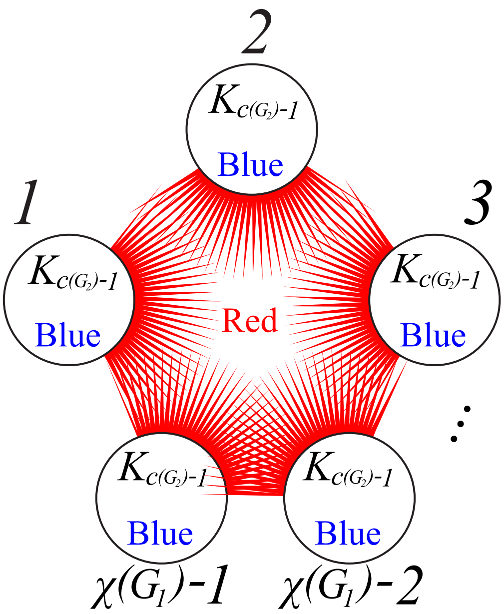


Figure 4.4: General graph Ramsey number lower bound

interconnecting edge red. If there were a red copy of G_1, then there would be a vertex colouring of G_1 with $\chi(G_1) - 1$ colours, producing no monochromatic edge, which is impossible. On the other hand, there can be no blue G_2, since the largest blue component has cardinality $c(G_2) - 1$. $\square$

Note that when $G_1 = K_\ell$ and $G_2 = K_k$, the construction used in the proof of Theorem 4.4.1 is exactly the first constructive bound given in Section 3.3.

Using the bound given by Theorem 4.4.1, Chvátal produced the precise value of the graph Ramsey numbers for complete graphs versus trees.

Theorem 4.4.2 (Chvátal, 1977 [25]). *For any $m, n \in \mathbb{Z}^+$ and any tree T on n vertices,*

$$R(K_m, T) = (m-1)(n-1) + 1.$$

Proof. The lower bound follows from Theorem 4.4.1. The upper bound is proved by induction on $m+n$. For any $k \geq 4$, let $S(k)$ be the statement that for all $m, n \in \mathbb{Z}^+$ such that $m+n = k$ and any tree T on n vertices, $R(K_m, T) \leq (m-1)(n-1)+1$.

Base Case: The statement $S(4)$ reduces to $R(K_2, T_2) \leq 2$, which is true, since $R(K_2, T_2) = R(2, 2) = 2$ (by Observation 3.1.4).

Inductive Step: Let $k \geq 5$, and assume that $S(k-1)$ holds. Let $m, n \in \mathbb{Z}^+$ be such that $m+n = k$, and let T be any tree on n vertices. It suffices to show that for any edge 2-colouring of $K_{(m-1)(n-1)+1}$, there is either a red K_m or a blue T.

Fix a colouring $\Delta : E(K_{(m-1)(n-1)+1}) \to \{\text{red}, \text{blue}\}$. Let x be any leaf node of T_n, and let $T' = T \setminus \{x\}$. Then, since $S(k-1)$ holds, $R(K_m, T') \leq (m-1)(n-2)+1$, and therefore either $K_{(m-1)(n-1)+1}$ contains a red K_m, in which case $S(k)$ holds, or a blue T', so assume there is a blue T'. Remove the $n-1$ points of the blue copy of T'. Then again since $S(k-1)$ holds, the remaining $K_{(m-2)(n-1)+1}$ either contains a red K_{m-1} or a blue T. If it has a blue T, again $S(k)$ holds, so assume it contains a red K_{m-1}. Therefore the original coloured $K_{(m-1)(n-1)+1}$ contains both a blue T' and a red K_{m-1}, disjoint from one-another. Let y be the end point in T' that x

was connected to, and consider the edges from y to the red K_{m-1}. If even one edge between y and the red K_{m-1} is coloured blue, a blue T is formed, and $S(k)$ holds. However, the alternative is that all edges between y and the red K_{m-1} are red, in which case a red K_m is formed, again showing that $S(k)$ holds.

Therefore by mathematical induction, for all $k \geq 4$, $S(k)$ holds. □

Recall (see Appendix) that for graphs H_1 and H_2, $H_1 \dot\cup H_2$ is the disjoint union of H_1 and H_2.

Theorem 4.4.3 (Burr, Erdős, and Spencer, 1975 [16]). *For all graphs G, H_1, and H_2,*

$$R(G, H_1 \dot\cup H_2) \leq max\{R(G, H_1) + |V(H_2)|, R(G, H_2)\}, \tag{4.2}$$

and further, for all $m, n \geq 1$,

$$R(mG, nH) \leq R(G, H) + (m-1)|V(G)| + (n-1)|V(H)|.$$

Proof. Let G, H_1, H_2 be graphs. Let $n = \max\{R(G, H_1) + |V(H_2)|, R(G, H_2)\}$, and 2-colour $E(K_n)$ with red and blue. Since $n \geq R(G, H_2)$, there is either a red G, in which case the proof is done, or a blue H_2. If there is a blue H_2, remove its vertices to be left with $n - |V(H_2)|$ vertices. However, $n - |V(H_2)| \geq R(G, H_1)$ and thus the remaining complete graph must contain either a red G, in which case again the proof is done, or a blue H_1, in which case there is a blue copy of $H_1 \dot\cup H_2$ in the original graph.

The second bound is proved by induction on $m+n$. For any $k \geq 2$, let $S(k)$ be the statement that for all $m, n \in \mathbb{Z}^+$ such that $m+n=k$, it holds that $R(mG, nH) \leq R(G, H) + (m-1)|V(G)| + (n-1)|V(H)|$.

Base Case: The statement $S(2)$ reduces to $R(G, H) \leq R(G, H)$, which is true.

Inductive Step: Let $k \geq 2$, and assume $S(k)$ holds. Let m, n be such that $m+n = k+1$. Since $k \geq 2$, one of m and n is at least two, without loss of generality, say $n \geq 2$. Then

$$\begin{aligned} R(mG, nH) &= R(mG, (n-1)H \,\dot{\cup}\, H) \\ &\leq \max\{R(mG, (n-1)H) + |V(H)|, R(mG, H)\} \qquad \text{(by (4.2))} \\ &= R(mG, (n-1)H) + |V(H)| \\ &\leq R(G, H) + (m-1)|V(G)| + (n-2)|V(H)| + |V(H)| \quad \text{(by } S(k)\text{)} \\ &= R(G, H) + (m-1)|V(G)| + (n-1)|V(H)|. \end{aligned}$$

Therefore $S(k+1)$ holds, and so by mathematical induction, for all $k \geq 2$, $S(k)$ holds. $\square$

Recall (see Appendix) that for any graphs G and H, $G+H$ is the graph formed by joining disjoint copies of G and H by every possible edge.

Theorem 4.4.4 (Burr and Erdős, 1973 [16]). *For any graphs G and H, and $n = R(G, H)$,*

$$R(G + K_1, H) \leq R(K_{1,n}, H).$$

Proof. Let G, H be graphs, $n = R(G, H)$, and let $N = R(K_{1,n}, H)$. Colour the edges of K_N with red and blue. If there is a blue H, the result holds, so assume there is none. Then by the definition of N, there is a red $K_{1,n}$, and by the definition of n, there is either a blue H, in which case the result again holds, or a red copy of G, which produces a red $G + K_1$. □

Here are some other representative theorems in the field, provided with references, but without proofs:

Theorem 4.4.5 (Parsons, 1973 [117]). *For $m, n \in \mathbb{Z}^+$, $R(P_m, K_n) = m(n-1) + 1$.*

Theorem 4.4.6 (Burr, 1974 [14]). *Let T_m be any tree on m vertices, and assume that $m - 1$ divides $n - 1$. Then $R(T_m, K_{1,n}) = m + n - 1$.*

Theorem 4.4.7 (Parsons, 1974 [118]). *If $n \geq m \geq 3$,*

$$R(P_m, K_{1,n}) = \max\{R(P_{m-1}, K_{1,n}), R(P_m, K_{1,n-m}) + m\}.$$

Theorem 4.4.8 (Burr, Erdős, and Spencer, 1975 [22]). *For $m, n \in \mathbb{Z}^+$,*

(a) *if $m \geq n \geq 1, m \geq 2$, then $R(mK_3, nK_3) = 3m + 2n$,*

(b) *if* $m \geq n, m \geq 2$, *then* $R(mK_{1,3}, nK_{1,3}) = 4m + n - 1$,

(c) $R(nG, nK_2) = (|V(G)| + 1)n - 1$,

(d) *if* $n \geq 2$, *then* $R(nK_m, nP_2) = (m + 2)n - 1$, *and*

(e) *if* $n \geq 2$, *and* G *is not complete, then* $R(nG, nP_2) = (|V(G)| + 1)n - 1$.

Theorem 4.4.9 (Burr, Erdős, Faudree, Rousseau, and Schelp, 1989 [19])**.** *For any tree* T *on* n *vertices, and maximum degree* d,

$$R(C_4, T) = \max\{4, n + 1, R(C_4, K_{1,d})\}.$$

When $k > 2$ or $r > 2$, very little work in comparison has been done for determining the values $R(G; K_k; r)$. The graph arrow notation, and off-diagonal graph Ramsey numbers $R(G_1, G_2)$ have a natural generalization to r colours, just as the Ramsey numbers $R(a, b)$ generalized to $R(m_1, \ldots, m_r)$. For any graphs $F, G_1, \ldots, G_r$, and H, write

$$F \longrightarrow (G_1, \ldots, G_r)_r^H$$

iff for any r-colouring $\Delta : \binom{F}{H} \to [1, r]$, there exists $i \in [1, r]$ and $G_i' \in \binom{F}{G_i}$ such that $\binom{G_i'}{H}$ is monochromatic in the i-th colour under Δ. In general, let $R(G_1, \ldots, G_r; H)$ denote the least integer n (if any exists) such that $K_n \longrightarrow (G_1, \ldots, G_r)_r^H$. For any $k, r \in \mathbb{Z}^+$, and any graphs $G_1, \ldots, G_r$, the values $R(G_1, \ldots, G_r; K_k)$ do exist since

$$R(G_1, \ldots, G_r; K_k) \leq R(G_1 \,\dot\cup \cdots \dot\cup\, G_r; K_k).$$

Chapter 5

Induced graph Ramsey theory

One might ask if there exists a graph F' such that $F' \longrightarrow (G)_r^{K_k}$ with the added condition that the found monochromatic copy of G must be an induced subgraph of F'. The main result of this chapter is that for any graph G and any $k, r \in \mathbb{Z}^+$, there exists such an F'.

5.1 Definitions

For any graphs G and H, denote the set of induced subgraphs of G isomorphic to H by

$$\binom{G}{H}_{\mathrm{ind}} = \{H' \preceq G : H' \cong H\}.$$

For any $r \in \mathbb{Z}^+$, and graphs G and H, write

$$F \xrightarrow{\text{ind}} (G)^H_r$$

iff for every $\Delta : \binom{F}{H}_{\text{ind}} \to [1, r]$, there exists $G' \in \binom{F}{G}_{\text{ind}}$ such that $\binom{G'}{H}_{\text{ind}}$ is monochromatic under Δ. For any $r \in \mathbb{Z}^+$, and any graphs G and H, let $R_{\text{ind}}(G; H; r)$ be the least integer n (if any exists) such that there exists a graph F on n vertices such that $F \xrightarrow{\text{ind}} (G)^H_r$. The numbers $R_{\text{ind}}(G; H; r)$ are called *induced graph Ramsey numbers*.

5.2 Relationship to weak graph Ramsey numbers

For all $k, m, r \in \mathbb{Z}^+$,

$$R_{\text{ind}}(K_m; K_k; r) = R(K_m; K_k; r) = R_k(m; r).$$

In general, the problem of determining for some $r \in \mathbb{Z}^+$, and graphs G and H whether the induced graph Ramsey number $R_{\text{ind}}(G; H; r)$ exists is different than determining if $R(G; H; r)$ exists. Consider the following two examples: First, $K_3 \longrightarrow (P_2)^{K_2}_2$, but $K_3 \overset{\text{ind}}{\not\longrightarrow} (P_2)^{K_2}_2$, and therefore a (weak) graph Ramsey arrow does not imply the corresponding induced graph Ramsey arrow. Second, for any $n \geq 3$, $P_2 \dot\cup K_n \xrightarrow{\text{ind}} (P_2 \dot\cup K_n)^{P_2}_2$, but $P_2 \dot\cup K_n \not\longrightarrow (P_2 \dot\cup K_n)^{P_2}_2$, and therefore an induced graph Ramsey arrow does not imply the corresponding (weak) graph Ramsey arrow. However, when H is complete (the most studied case), the induced Ramsey arrow is stronger than the weak Ramsey arrow ($F \xrightarrow{\text{ind}} (G)_r^{K_k}$ implies that $F \longrightarrow (G)_r^{K_k}$).

In the weak case, it is true that for any graphs G and G' such that $G' \subseteq G$, $R(G'; K_k; r) \leq R(G; K_k; r)$, and therefore for any graph G on m vertices, $R(G; K_k; r) \leq R(K_m; K_k; r)$, which exists by Ramsey's theorem (as mentioned in the introduction, p. 8). The analogous statement for induced subgraphs does hold.

Observation 5.2.1. *For any graphs G and G' such that $G' \preceq G$, if $R_{ind}(G; K_k; r)$ exists, then $R_{ind}(G'; K_k; r) \leq R_{ind}(G; K_k; r)$.*

However, if $G' \subseteq G$, then no such relation holds, as exhibited by the following observation (which is likely folklore, but I couldn't find it in the literature).

Observation 5.2.2. *If $R_{ind}(P_2; K_1; 2)$ exists, then $R_{ind}(P_2; K_1; 2) > R_{ind}(K_3; K_1; 2)$.*

Proof. By the pigeonhole principle, $R_{\text{ind}}(K_3; K_1; 2) = 5$. Let F be any graph on five vertices. It remains to show that there exists a colouring of the vertices of F such that every induced copy of P_2 is not monochromatic.

Claim. There are three vertices $I = \{x, y, z\} \subseteq V(F)$ such that the induced subgraph on I, $F[I]$, is not isomorphic to P_2.

Proof of Claim. If F is bipartite, then let I be any set of three independent vertices. If F is not bipartite, then F contains an odd cycle. If F contains a C_3, then let I be the vertices of this C_3. Otherwise, F is exactly the graph C_5, which contains such a triple. This proves the claim.

Let $\Delta : V(F) \to \{\text{red}, \text{blue}\}$ be defined by $\Delta(x) = \Delta(y) = \Delta(z) = \text{red}$, and the other two vertices both blue. $\square$

The same colour grouping argument used to prove Proposition 5.2.3 can be used to prove the analogous theorem for the induced arrow. The proof is omitted.

Proposition 5.2.3 (Ramsey, 1930 [126]). *Let $k \in \mathbb{Z}^+$. If for any graph G, the induced Ramsey number $R_{ind}(G; K_k; 2)$ exists, then for any $r > 2$, it holds that for any graph G, $R_{ind}(G; K_k; r)$ exists as well.*

5.3 Off-diagonal generalization

Similar to the Ramsey numbers in Chapter 3, and the graph Ramsey numbers in Chapter 4, one can generalize the induced graph Ramsey arrow and these induced graph Ramsey numbers to off-diagonal versions. For any graphs F, H, G_1, and G_2, write

$$F \xrightarrow{\text{ind}} (G_1, G_2)_2^H$$

iff for every $\Delta : \binom{F}{H}_{\text{ind}} \to \{\text{red}, \text{blue}\}$, there either exists $G_1' \in \binom{F}{G_1}_{\text{ind}}$ such that $\binom{G_1'}{H}_{\text{ind}}$ is monochromatic red, or there exists $G_2' \in \binom{F}{G_2}_{\text{ind}}$ such that $\binom{G_2'}{H}_{\text{ind}}$ is monochromatic blue. Let $R_{\text{ind}}(G_1, G_2)$ denote the least integer n (if any exists) such that there exists a graph F on n vertices such that $F \xrightarrow{\text{ind}} (G_1, G_2)_2^{K_2}$. Note that $R_{\text{ind}}(G_1, G_2) \leq R_{\text{ind}}(G_1 \dot{\cup} G_2; K_2; 2)$, and therefore proving the existence of the diagonal version implies the off-diagonal.

Very few results bounding induced Ramsey numbers are known, but the interested reader can find such bounds in, *e.g.*, [58, 59, 60, 73, 83, 88, 93].

5.4 Colouring vertices ($H = K_1$)

For any two graphs G and H, the *lexicographic product of* G *and* H, denoted $G \otimes H$, is the graph with vertices $V(G) \times V(H)$, and edge set E, where

$$\{(u_1, v_1), (u_2, v_2)\} \in E \text{ iff } \begin{cases} \{u_1, u_2\} \in E(G), \text{ or} \\ u_1 = u_2 \text{ and } \{v_1, v_2\} \in E(H). \end{cases}$$

By the definition, note that "$\otimes$" is not commutative.

Note that this notation "$\otimes$" is certainly non-standard, and that this lexicographic product is one of many different types of products on graphs. For more information on this and other types (*e.g.*, direct product, cartesian product), see [76].

Folkman showed that for any graph G, $R_{\text{ind}}(G; K_1; 2)$ exists.

Theorem 5.4.1 (Folkman, 1970 [53])**.** *For any graph G,*

$$G \otimes G \xrightarrow{ind} (G)_2^{K_1}.$$

Proof. Let G be a graph, and let $F = G \otimes G$. Let $\Delta : V(F) \to \{\text{red}, \text{blue}\}$ be a $\{\text{red}, \text{blue}\}$-colouring of $V(F)$. If there exists a vertex $x_0 \in V(G)$ such that for all $y \in V(G)$, (x_0, y) is coloured red, then there is an induced red copy of G on the vertices $\{(x_0, y) : y \in V(G)\}$, and the theorem is proved. Otherwise, for every vertex $x \in V(G)$, there exists $y = y(x) \in V(G)$ such that $(x, y(x))$ is coloured blue. Then there is an induced blue copy of G on the vertices $\{(x, y(x)) : x \in V(G)\}$. $\square$

5.5 Colouring edges ($H = K_2$)

The problem of determining whether or not, for any graph G, there exists a graph F such that $F \xrightarrow{\text{ind}} (G)_2^{K_2}$ was also solved. The proofs of the existence of such an F were first published independently by Deuber in 1975 [32], Erdős, Hajnal, and Pósa in 1975 [42], and Rödl in 1976 [130]. The first proof presented below is based on Deuber's [32] original proof, as given in [33] (Section 5.5.1), which is an inductive construction. The second proof presented is due to Nešetřil and Rödl in 1981 [109] (Section 5.5.3), and uses a technique called "partite amalgamation" (which is covered in Section 5.5.2).

5.5.1 Deuber's proof

First some notation is needed. For $q \in \mathbb{Z}^+$, and f a function, f^q denotes the *q-fold composition of f*, $\underbrace{f \circ f \circ \cdots \circ f}_{q}$, and let f^0 denote the identity function.

For any graphs G and H, and any induced subgraph $U \preceq G$, the *generalized lexicographic product of G and H over U*, denoted $G \otimes_U H$, is the graph with vertices $V(U \otimes H) \cup ((V(G) \setminus V(U)) \times \{\emptyset\})$ and edge set E where

$$\{(u_1, v_1), (u_2, v_2)\} \in E \text{ iff } \begin{cases} \{(u_1, v_1), (u_2, v_2)\} \in E(U \otimes H), \text{ or} \\ v_1 = \emptyset \text{ and } \{u_1, u_2\} \in E(G). \end{cases}$$

Note Figure 5.1, and other figures in this section, are based on those found in [33].

Deuber proved the following off-diagonal theorem.

Theorem 5.5.1 (Deuber, 1975 [32]). *For any graphs G_1, G_2, there exists a graph F such that $F \xrightarrow{ind} (G_1, G_2)_2^{K_2}$.*

Proof. This proof is found in Diestel's "Graph Theory" [33, pp. 259–262] with only notational changes. The proof is by strong induction on $|V(G_1)| + |V(G_2)|$.

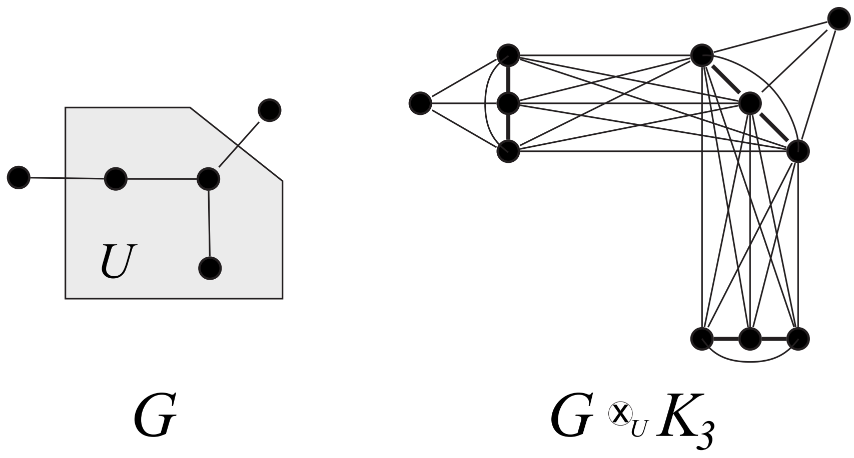


Figure 5.1: A generalized lexicographic product

For any $n \in \mathbb{Z}^+$, let $S(n)$ be the statement that for any graphs G_1 and G_2 such that $|V(G_1)| + |V(G_2)| = n$, there exists an F such that $F \xrightarrow{\text{ind}} (G_1, G_2)_2^{K_2}$.

Base Case: If either of G_1 or G_2 contain no edges (which includes the case when $|V(G_1)| + |V(G_2)| \leq 2$), then for $k = \max\{|V(G_1)|, |V(G_2)|\}$, $\overline{K_k} \xrightarrow{\text{ind}} (G_1, G_2)_2^{K_2}$. Therefore $S(2)$ holds.

Inductive Step: Let $k \geq 2$, and assume that $S(2), \ldots, S(k)$ all hold. Let G_1, G_2 be graphs, each with at least one edge, such that $|V(G_1)| + |V(G_2)| = k + 1$. For each $\ell \in \{1, 2\}$, fix $x_\ell \in V(G_\ell)$, incident with at least one edge, let $G'_\ell = G_\ell \backslash \{x_\ell\}$, and let $G''_\ell = G_\ell[N(x_\ell)]$, the graph induced in G_ℓ on the vertex set $N(x_\ell)$ (see Figure 5.2).

By $S(k)$, there exists graphs F_1 and F_2 such that $F_1 \xrightarrow{\text{ind}} (G_1, G'_2)_2^{K_2}$ and $F_2 \xrightarrow{\text{ind}}$

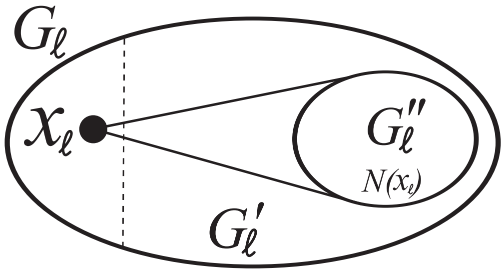


Figure 5.2: The graphs G_ℓ, G'_ℓ, and G''_ℓ.

$(G'_1, G_2)_2^{K_2}$. A sequence of graphs $\Gamma_1, \Gamma_2, \ldots, \Gamma_{n+1}$ is now constructed such that

$$\Gamma_{n+1} \xrightarrow{\text{ind}} (G_1, G_2)_2^{K_2}.$$

Let $\Gamma_1 = F_1$. Let $n = \left|\binom{\Gamma_1}{G'_2}_{\text{ind}}\right|$ and enumerate $\binom{\Gamma_1}{G'_2}_{\text{ind}} = \{W'_1, W'_2, \ldots, W'_n\}$. For each $i \in [1, n]$, fix $W''_i \in \binom{W'_i}{G''_2}_{\text{ind}}$. Let $V_1 = V(\Gamma_1)$.

For the inductive step of the construction, let $i \in [1, n]$, and assume that $\Gamma_1, \ldots, \Gamma_i$ and $V_1, \ldots, V_i$ have all been defined. Further, if $i > 1$, assume that a function $f : V_2 \cup \cdots \cup V_i \to V_1 \cup \cdots \cup V_{i-1}$ has been defined such that for all $j \in [2, i]$, $f(V_j) = V_{j-1}$. Note that $f^{q-1}(V_q) = V_1$. Let $U_i = \{v \in V_i : f^{i-1}(v) \in W''_i\}$ (then $U_0 = W''_1$).

Defining Γ_{i+1} is done in two steps. First, let $\hat{\Gamma}_{i+1} = \Gamma_i \otimes_{\Gamma_i[U_i]} F_2$, and define $V_{i+1} \subseteq V(\hat{\Gamma}_{i+1})$ as $V_{i+1} = \{(a, b) \in V(\hat{\Gamma}_{i+1}) : a \in V_i\}$. For each $u \in U_i$, let $F_2(u)$ denote the copy of F_2 in $\hat{\Gamma}_{i+1}$ that replaces u. Enumerate $U_i = \{u_1, \ldots, u_m\}$.

Second, let

$$\mathcal{C} = \left\{ (G_1'(u_1), G_1'(u_2), \ldots, G_1'(u_m)) : G_1'(u_j) \in \binom{F_2(u_j)}{G_1'}_{\text{ind}} \right\}.$$

For each $C = (G_1'(u_1), \ldots, G_1'(u_m)) \in \mathcal{C}$, add a new vertex $x(C)$ to $\hat{\Gamma}_{i+1}$. For each $u \in U_i$, fix

$$G_1''(u) \in \binom{G_1'(u)}{G_1''}_{\text{ind}}.$$

Finally, join $x(C)$ to every vertex in the set $\{V(G_1''(u_1)) \cup \cdots \cup V(G_1''(u_m))\}$. The resulting graph is Γ_{i+1} (see Figure 5.3).

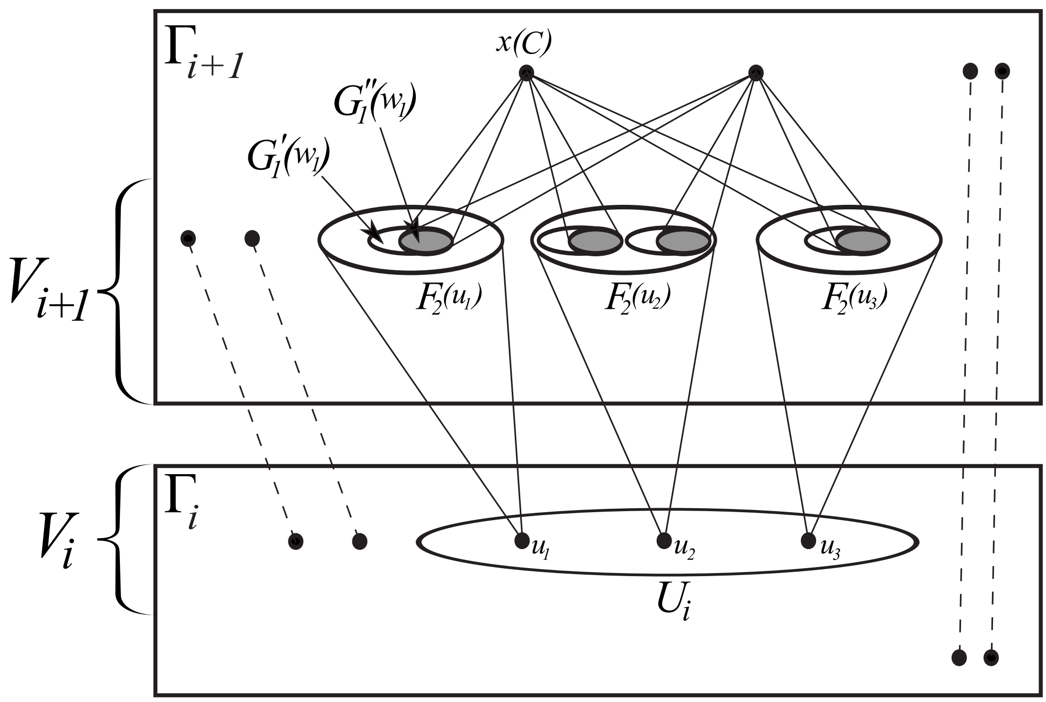


Figure 5.3: The inductive construction: $\Gamma_i \to \Gamma_{i+1}$

Extend $f : V_2 \cup \cdots \cup V_i \to V_1 \cup \cdots \cup V_{i-1}$ to the function $\hat{f} : V_2 \cup \cdots \cup V_{i+1} \to V_1 \cup \cdots \cup V_i$

by defining for each $v \in V_{i+1}$,

$$\hat{f}(v) = \begin{cases} u \text{ if } v \in F_2(u), u \in U_i, \text{ or} \\ \\ v' \text{ if } v = (v', \emptyset), v' \in V_i \setminus U_i. \end{cases}$$

This completes the inductive definition of $\Gamma_1, \Gamma_2, \ldots, \Gamma_{n+1}$.

Claim. For all $i \in [1, n+1]$, and for any red-blue colouring of the edges of Γ_i, there exists either (a) a red induced copy of G_1, (b) a blue induced copy of G_2, or (c) a blue induced Λ, where $V(\Lambda) \subseteq V_i$ and there exists $k \in \{i, \ldots, n\}$ such that the restriction of f^{i-1} to $V(\Lambda)$ is an isomorphism between Λ and W'_k.

(Note that case (c) cannot occur when $i = n+1$, and therefore proving the claim proves that $\Gamma_{n+1} \xrightarrow{\text{ind}} (G_1, G_2)_2^{K_2}$.)

Proof of Claim. The proof of the claim is by strong induction on i. For any $j \in [1, n+1]$, let $T(j)$ be the statement that for any red-blue colouring of the edges of Γ_j, one of (a), (b), or (c) holds.

Base Case: Let $\Delta : E(\Gamma_1) \to \{\text{red}, \text{blue}\}$. Then since $\Gamma_1 = F_1$, and $F_1 \xrightarrow{\text{ind}} (G_1, G'_2)_2^{K_2}$, there is either an red induced copy of G_1, and (a) holds, or there is a blue copy of G'_2, and since any copy of G'_2 is one of the W'_k's by definition, (c) holds. Therefore $T(1)$ is true.

Inductive Step: Let $i \in [1, n]$, and assume that $T(1), \ldots, T(i)$ all hold. Let Δ :

$E(\Gamma_{i+1}) \to \{\text{red}, \text{blue}\}$. For each $u \in U_i$, there exists $F_2(u) \in \binom{\Gamma_{i+1}}{F_2}_{\text{ind}}$. Since $F_2 \xrightarrow{\text{ind}} (G_1' G_2)_2^{K_2}$, each copy of F_2 either contains a red copy of G_1' or a blue copy of G_2. If there exists $u_0 \in U_i$ such that $F_2(u_0)$ contains a blue induced copy of G_2, then (b) holds, so assume for all $u \in U_i$, $F_2(u)$ does not contain a blue induced copy of G_2. Then for all $u \in U_i$, $F_2(u)$ contains a red induced copy of G_1', call it $G_1'(u)$.

Enumerate $U_i = \{u_1, \ldots, u_m\}$, and let $C = (G_1'(u_1), \ldots, G_1'(u_m))$. Then for every $u \in U_i$, there exists

$$G_1''(u) \in \binom{G_1'(u)}{G_1''}_{\text{ind}}$$

and $x = x(C) \in V(\Gamma_{i+1})$ such that x is connected to every vertex in the set $\bigcup_{u \in U_i} V(G_1''(u))$. If for any particular $u_0 \in U_i$ all the edges

$$\{\{x, v\} : v \in V(G_1''(u_0))\}$$

are red, then $G_1'(u_0)$ together with x forms a red G_1, and (a) holds. Otherwise, for every $u \in U_i$, there exists $y_u \in V(G_1''(u))$ such that the edge $\{x, y_u\}$ is blue. Let $D = \{y_u : u \in U_i\} \cup \{(v, \emptyset) : v \in V(\Gamma_i) \setminus U_i\}$. Then f restricted to D, $f|_D$, is an isomorphism between $\Gamma_{i+1}[D]$ and Γ_i. Therefore the 2-colouring on the edges of $\Gamma_{i+1}[D]$ induces a 2-colouring on the edges of Γ_i. By $T(i-1)$, one of (a), (b), or (c) holds for Γ_i. If (a) or (b) holds for Γ_i, then using $f|_D$, the same holds in Γ_{i+1}, so assume (c) holds in Γ_i. Then there exists an induced blue $\Lambda \preceq \Gamma_i$ such that $V(\Lambda) \subseteq V_i$, and there exists $k \in \{i, \ldots, n\}$ such that the restriction of f^{i-1} to $V(\Lambda)$

is an isomorphism between Λ and W'_k.

If $k \in \{i, \ldots, n\}$, then $\hat{\Lambda} = (f|_D)^{-1}(\Lambda)$ satisfies (c) in Γ_{i+1}.

Otherwise $k = i - 1$. Then the restriction of f^{i-1} to $V(\Lambda)$ is an isomorphism between Λ and W'_{i-1}. Since Λ is a copy of W'_{i-1}, let $\Lambda'' \preceq \Lambda$ be defined as the induced subgraph with $V(\Lambda'') = \{v \in V(\Lambda) : f^{i-1}(v) \in W''_i\}$ (Λ'' is a copy of W''_i). Then by the definition of U_i, $V(\Lambda'') \subseteq U_i$ (see Figure 5.4).

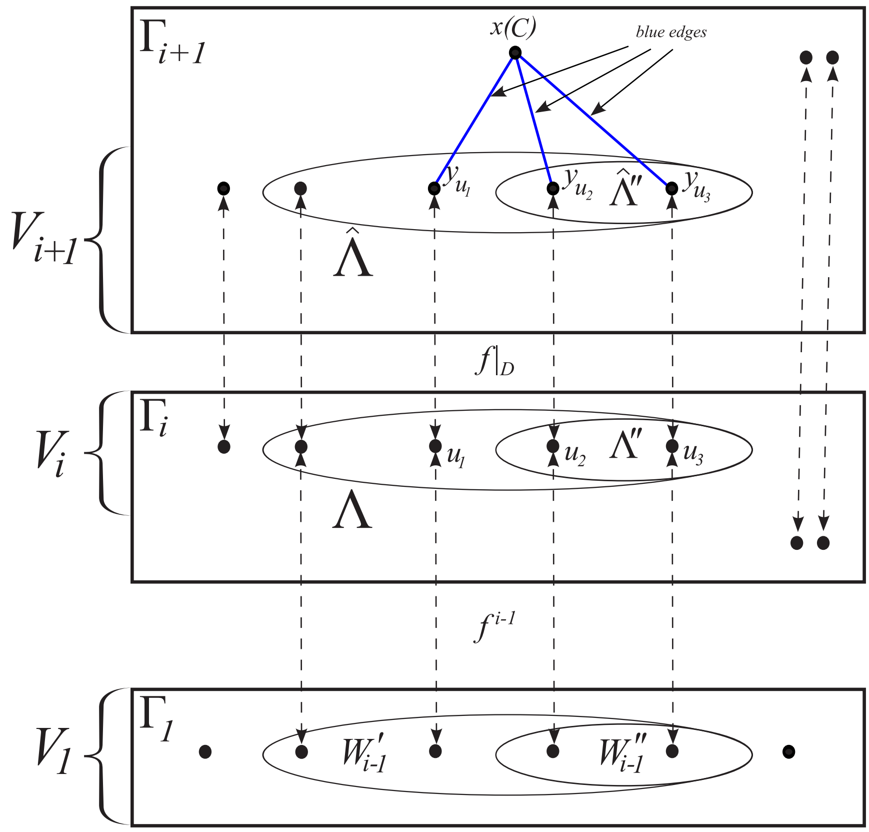


Figure 5.4: Producing a monochromatic copy of G_2

Then $\hat{\Lambda}'' = (f|_D)^{-1}(\Lambda'') \in \binom{\hat{\Lambda}}{W''_{i-1}}$ is such that $V(\hat{\Lambda}'') \subseteq \{y_u : u \in U_i\}$. Therefore each vertex in $\hat{\Lambda}''$ is attached to x by a blue edge. Since Λ itself was blue, so is all of $\hat{\Lambda}$. Therefore $\hat{\Lambda}$ together with x forms a blue copy of G_2. Therefore $T(i+1)$ holds.

Therefore by induction, for all $j \in [1, n+1]$, $T(j)$ holds, which in turn proves the theorem. □

5.5.2 Partite Amalgamation

The partite amalgamation (or simply amalgamation) process was developed by Nešetřil and Rödl, and, according to Nešetřil, "originated in 1976" [102, p. 1389]). Various versions of amalgamation have been used to prove a number of theorems in a constructive manner (see *e.g.* [102, 108, 109, 110, 111, 113, 114, 115]). What follows is one type of amalgamation, known as $*_J$-amalgamation (for graphs).

Recall (see Appendix) that for any $k \in \mathbb{Z}^+$, a graph G is *k-partite* iff there exists a partition of $V(G) = V_1 \cup \cdots \cup V_k$ such that for each edge $\{x, y\} \in E(G)$, there exists $i, j \in [1, k]$, $i \neq j$, such that $x \in V_i$ and $y \in V_j$. To emphasize that a graph G is k-partite, one can write $G = (V_1 \cup \cdots \cup V_k, E)$. A 2-partite graph is called *bipartite*.

When focusing specifically on partite graphs, a notation analogous to that for induced subgraphs is used: let $G = (V_1 \cup \cdots \cup V_n, E(G))$ be an n-partite graph, and let $H = (W_1 \cup \cdots \cup W_m, E(H))$ be an m-partite graph. Then H is an *induced m-partite* *subgraph of* G, written $H \preceq_{\text{part}} G$, iff there exists an injection $f : [1, m] \to [1, n]$ such that for all $i \in [1, m]$, $W_i \subseteq V_{f(i)}$, and $E(H) = E(G) \cap [W_1 \cup \cdots \cup W_m]^2$. The set of partite induced subgraphs of G that are isomorphic to H is denoted

$$\binom{G}{H}_{\text{part}} = \{H' \preceq_{\text{part}} G : H' \cong H\}.$$

Perhaps it would be more consistent to denote this instead as "$\binom{G}{H}_{\substack{\text{part}\\\text{ind}}}$", but since weak partite subgraphs are not of interest here, the "ind" can be dropped.

Let $a, b \in \mathbb{Z}^+$ with $a \leq b$, let $A = (X_1 \cup \cdots \cup X_a, \mathcal{A})$ be an a-partite graph, and let $B = (Y_1 \cup \cdots \cup Y_b, \mathcal{B})$ be a b-partite graph. Let $Y = \cup_{i=1}^{b} Y_i$. Fix $J \in [1, b]^a$, and fix an order preserving bijection $f : J \to [1, a]$. The $*_J$*-amalgamation of* B *with* A,

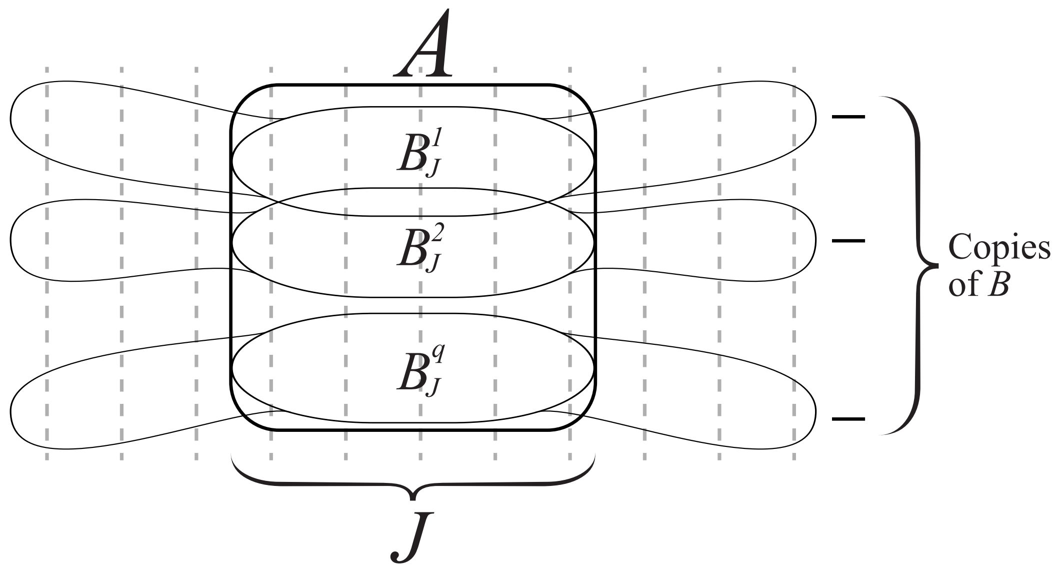


Figure 5.5: $*_J$-amalgamation

denoted $B *_J A$, is defined as follows. Let B_J be the induced subgraph of B on the vertices $\cup_{j\in J} Y_j$, *i.e.*,

$$B_J = \left(\bigcup_{j\in J} Y_j,\ \mathcal{B} \cap \left[\bigcup_{j\in J} Y_j \right]^2 \right).$$

Let $\left|\binom{A}{B_J}_{\text{part}}\right| = q$, and enumerate $\binom{A}{B_J}_{\text{part}} = \{B_J^1, B_J^2, \ldots, B_J^q\}$. For all $i \in [1, b]$, define

$$Z_i = \begin{cases} X_{f(i)} & \text{if } i \in J, \text{ and} \\ Y_i \times [1, q] & \text{otherwise.} \end{cases}$$

For each $j \in [1, q]$, fix an isomorphism $\phi_j : B_J \to B_J^j$, and extend ϕ_j to the function $\psi_j : Y \to \bigcup_{i=1}^{b} Z_i$ by

$$\psi_j(y) = \begin{cases} \phi_j(y) & \text{if } y \in B_J, \text{ and} \\ (y, j) & \text{otherwise.} \end{cases}$$

Define the edge set $\mathcal{E}$ as $\mathcal{E} = \{\{\psi_j(v_1), \psi_j(v_2)\} : \{v_1, v_2\} \in E(B), j \in [1, q]\} \cup E(A)$. Finally, the $*_J$-amalgamation $B *_J A$ is the b-partite graph $(Z_1 \cup \cdots \cup Z_b, \mathcal{E})$ (see Figure 5.5).

5.5.3 Nešetřil and Rödl's proof

For any $d, m \in \mathbb{Z}^+$, any d-set X, and $Y = [X]^m$, let $B(d, m)$ denote the bipartite graph $(X \cup Y, E)$, where $\{x, A\} \in E$ iff $x \in A$.

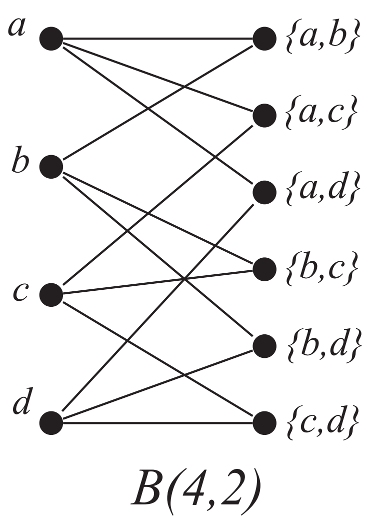


Figure 5.6: A sample graph of the form $B(d, m)$.

Nešetřil and Rödl's proof that for every graph G, $R_{\text{ind}}(G; K_2; 2)$ exists uses the following lemma:

Lemma 5.5.2 (Nešetřil and Rödl 1981 [109]). *For any bipartite graph G, there exists $d, m \in \mathbb{Z}^+$ such that $G \preceq B(d, m)$.*

Proof. Let $G = (V_1 \cup V_2, E)$ be a bipartite graph, and let $k_1 = |V_1|$, $k_2 = |V_2|$. Enumerate $V_1 = \{v_1, v_2, \ldots, v_{k_1}\}$, $V_2 = \{w_1, w_2, \ldots, w_{k_2}\}$, and let $Q = \{q_1, \ldots q_{k_1}\}$ be a set of k_1 elements, disjoint from both V_1 and V_2. It suffices to prove the following claim:

Claim: $G \preceq B(|V_1| + |V_2| + |Q|, k_1 + 1) = B(k_2 + 2k_1, k_1 + 1)$.

Let $X = V_1 \cup V_2 \cup Q$, $Y = [X]^{k_1+1}$. For any $w \in V_2$, let $\ell(w) = |V_1 \setminus N(w)|$, let $Q(w)$ be any set in $[Q]^\ell$, and define $S(w) = N(w) \cup \{w\} \cup Q(w)$. Then $|S(w)| =$

$|V_1|+1 = k_1+1$, and therefore $S(w) \in Y$. If $S(w_1) = S(w_2)$, then $w_1 = S(w_1) \cap V_2 = S(w_2) \cap V_2 = w_2$. Therefore if $w_1 \neq w_2$, $S(w_1) \neq S(w_2)$. Let $f : V(G) \to X \cup Y$ be defined by $f(v) = v$ if $v \in V_1$, and $f(v) = S(v)$ if $v \in V_2$. Then f is an embedding of G into an induced subgraph of $B(k_2 + 2k_1, k_1 + 1)$. □

For any $k \in \mathbb{Z}^+$, and k-partite graphs F and G, write

$$F \xrightarrow{\text{ind}}_{\text{part}} (G)_2^{K_2}$$

iff for every $\Delta : E(F) \to \{\text{red}, \text{blue}\}$, there exists $G' \in \binom{F}{G}_{\text{part}}$ such that $E(G')$ is monochromatic under Δ.

Theorem 5.5.3 (Bipartite Lemma, Nešetřil and Rödl, 1981 [109].)**.** *For all $r \in \mathbb{Z}^+$, and for every bipartite graph B, there exists a bipartite graph R_B such that*

$$R_B \xrightarrow{\textit{ind}}_{\textit{part}} (B)_r^{K_2}.$$

Proof. This proof is based on that given in [68]. By Lemma 5.5.2, assume without loss of generality that for some $d, m \in \mathbb{Z}^+$, $B = B(d, m)$. Let $\ell = r(m-1)+1$ and $n = R_\ell(d\ell; r\binom{\ell}{m})$. Let $R_B = B(n, \ell)$. To prove Theorem 5.5.3, it suffices to prove that

$$R_B \xrightarrow{\text{ind}}_{\text{part}} (B)_r^{K_2}.$$

Let $\Delta : E(R_B) \to [1, r]$. Assume the partite sets of $B(n, \ell)$ are $X = \{x_1, \ldots, x_n\}$ and $Y = [X]^\ell$. For each $S = \{s_1, \ldots, s_\ell\} \in Y$ (ordered respecting the ordering of

X), define $\Delta_S : S \to [1,r]$ by $\Delta_S(s) = \Delta(\{s,S\})$. Since $|S| = \ell = (m-1)r+1$, the pigeonhole principle implies that there exists $\beta(S) = \{\beta_1,\ldots,\beta_m\} \in [1,\ell]^m$ and $c_s \in [1,r]$ such that $C(S) = \{s_{\beta_1},\ldots,s_{\beta_m}\} \in [S]^m$ is monochromatic under Δ_S in the colour c_s. That is, for all $x \in C(S)$, $\Delta(\{x,S\}) = c_s$ (see Figure 5.7).

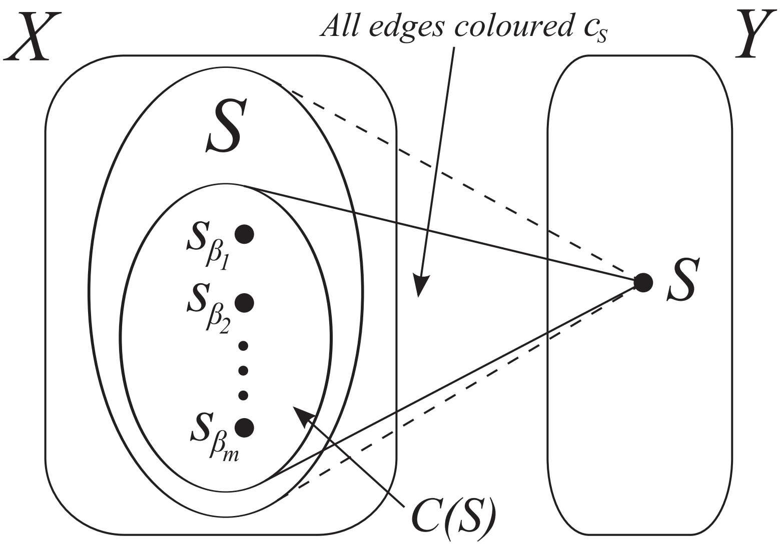


Figure 5.7: The set $C(S)$

Define $\Delta' : Y \to [1,r] \times [1,\ell]^m$ by $\Delta'(S) = (c_S, \beta(S))$. Since $Y = [X]^\ell$, and $|X| = n = R_\ell\left(d\ell; r\binom{\ell}{m}\right)$, there exists $R \in [X]^{d\ell}$ such that Δ' is constant on $[R]^\ell$, *i.e.*, there exists $c \in [1,r]$, $\beta = \{\beta_1,\ldots,\beta_m\} \in [1,\ell]^m$ such that for all $S = \{s_1,\ldots,s_\ell\} \in [R]^\ell$ (respecting the ordering of X), $C(S) = \{s_{\beta_1},\ldots,s_{\beta_m}\}$ is such that for all $x \in C(S)$, $\Delta(\{x,S\}) = c$ (see Figure 5.8).

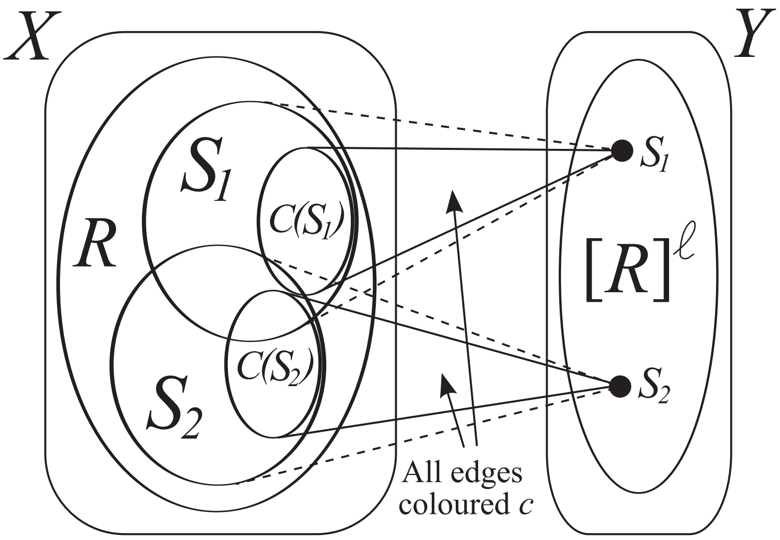


Figure 5.8: The set R

Respecting the ordering on X, enumerate $R = \{r_1, \ldots, r_{d\ell}\}$, and let

$$R' = \{r_{\beta_1}, r_{\beta_1+\ell}, \ldots, r_{\beta_1+(d-1)\ell}\};$$

note that since $\beta_1 \leq \ell$, it follows that $\beta_1 + (d-1)\ell \leq d\ell = |R|$, and thus R' is well defined.

Informally, for each $A \in [R']^m$, let $S_A \in [R]^\ell$ be such that $S_A \cap R' = A$, and the elements of A are positioned in positions $\beta = \{\beta_1, \ldots, \beta_m\}$ inside S_A. Formally, for each $A = \{a_1, \ldots, a_m\} \in [R']^m$, let $S_A = \{b_1, \ldots, b_\ell\} \in [R]^\ell$ (ordered respecting the ordering of X) be such that for each $i \in [1, m]$, $b_{\beta_i} = a_i$, and for each $j \in [1, \ell] \setminus \beta$, $b_j \in R \setminus R'$ (see Figure 5.9).

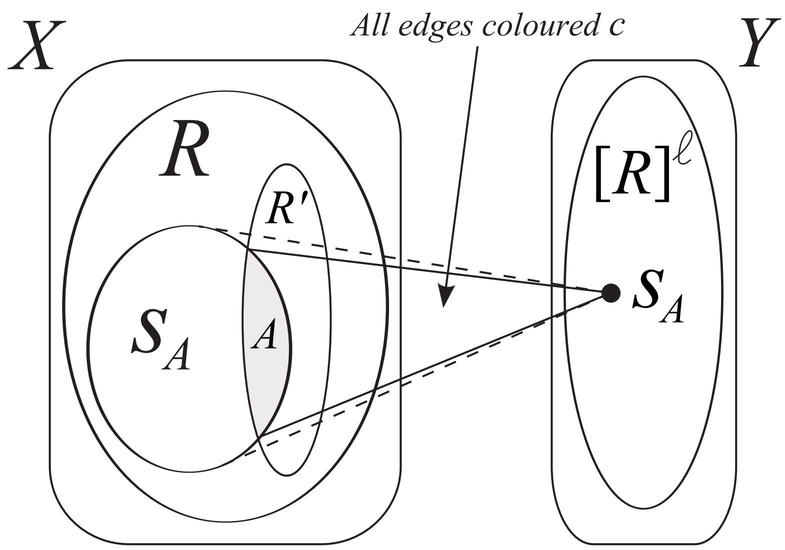


Figure 5.9: Definition of S_A

The the induced subgraph of R_B on the partite sets

$$R' \subseteq X \qquad \text{and} \qquad \{S_A : A \in [R']^m\} \subseteq Y$$

is a copy of $B(d, m)$, monochromatic in colour c. This completes the proof. □

Theorem 5.5.4 (Nešetřil and Rödl, 1981 [109]). *For all $r \in \mathbb{Z}^+$, and for any graph G, there exists a graph F such that $F \xrightarrow{ind} (G)_r^{K_2}$.*

Proof. A sequence of graphs $P^0, P^1, \ldots, P^{\binom{s}{2}}$ is constructed inductively, and the graph $F = P^{\binom{s}{2}}$ is the required graph.

Let G be a finite graph, and fix $r \in \mathbb{Z}^+$. Let $m = |V(G)|$, and by Ramsey's theorem, let $s = R_2(m; r)$. Let $H = \{1, 2, \ldots, s\}$. Enumerate $[H]^2 = \{e_1, e_2, \ldots, e_{\binom{s}{2}}\}$ and

$[H]^m = \{A_1, A_2, \ldots, A_{\binom{s}{m}}\}$. For all $i \in \{1, 2, \ldots, s\}$, define the set V_i^0 as

$$V_i^0 = \{i\} \times \left\{1, 2, \ldots, \binom{s}{m}\right\}.$$

For each $j \in \{1, 2, \ldots, \binom{s}{m}\}$, fix a bijection $f_j : V(G) \longrightarrow A_j \times \{j\}$. Define $\mathcal{E}$ as

$$\mathcal{E} = \left\{\{(i,j),(i',j)\} : \{f_j^{-1}(i,j), f_j^{-1}(i',j)\} \in E(G), j = 1, 2, \ldots, \binom{s}{m}\right\}.$$

Then let P^0 be the s-partite graph $P^0 = (V_1^0 \cup V_2^0 \cup \cdots \cup V_s^0, \mathcal{E})$. In effect, P^0 is a graph with one copy of G across each row, each copy drawn on a different collection of m vertices.

Let $k \geq 0$, and assume that $P^k = (V_1^k \cup V_2^k \cup \ldots \cup V_s^k, \mathcal{E}^k)$ has been defined. Assume $e_{k+1} = \{\alpha_1, \alpha_2\}$, and let B be the induced bipartite subgraph of P^k on the vertices $V_{\alpha_1}^k \cup V_{\alpha_2}^k$. Then by Theorem 5.5.3, there exists a bipartite graph, call it R_B, such that $R_B \xrightarrow{\text{ind}}_{\text{part}} (B)_r^{K_2}$. Then let $J = \{\alpha_1, \alpha_2\}$, define $P^{k+1} = P^k *_J R_B$. This completes the inductive construction.

Colour the edges of $P^{\binom{s}{2}}$ with r colours. Assume $e_{\binom{s}{2}} = \{\beta_1, \beta_2\}$. By construction, there is a copy of $P^{\binom{s}{2}-1}$, call it $P_*^{\binom{s}{2}-1}$, inside $P^{\binom{s}{2}}$, with all edges between the partitions $V_{\beta_1}^{\binom{s}{2}-1}$ and $V_{\beta_2}^{\binom{s}{2}-1}$ the same colour. Then again by construction, there is a copy of $P^{\binom{s}{2}-2}$, call it $P_*^{\binom{s}{2}-2}$, inside $P_*^{\binom{s}{2}-1}$, with all edges between two other partitions also the same colour. Continue until a copy of P^0, is found, and call it P_*^0. Then P_*^0 has the property that the colour of each edge is determined by the pair of partitions it connects. Define an r-colouring Δ on the set $[1, s]^2$ where

$\Delta(\{a,b\})$ is the colour of all edges between the a^{th} and b^{th} partitions. By the choice of s, there exists a monochromatic m-set $M = \{c_1, \ldots, c_m\} \in [1,s]^m$ such that all its pairs have the same colour. The subgraph of P^0_* on the partite sets $V^0_{c_1}, \ldots, V^0_{c_m}$ has all its edges the same colour. By the construction of P^0, the subgraph on the partite sets $V^0_{c_1}, \ldots, V^0_{c_m}$ contains exactly one induced copy of G. $\square$

Using the same proof technique, Nešetřil and Rödl [114] generalized Theorem 5.5.4 to colouring edges in k-uniform hypergraphs, and even more general structures known as systems.

5.6 Parameter words

Nešetřil and Rödl proved that for any graph G, and any $k \in \mathbb{Z}^+$, there exists a graph F such that $F \xrightarrow{\text{ind}} (G)^{K_k}_2$ using amalgamation techniques (see, *e.g.*, [102, 121]), but another technique, "parameter sets", is used here. For more information on parameter sets, see [122].

5.6.1 Definitions

An *alphabet* A is a finite set whose elements are called symbols. For any $n \in \mathbb{Z}^+$, define $A^n = \{f : [1,n] \to A\}$. Each function $f \in A^n$ can be viewed as the ordered n-tuple $(f(1), f(2), \ldots, f(n))$ of elements of A. The elements of A^n are also called *words of length n*

Let $\lambda_1, \lambda_2, \ldots$ denote symbols not in A, called *parameters*. For any $n \in \mathbb{Z}^+, m \in \mathbb{Z}^+ \cup \{0\}$, an *$m$-parameter word of length n over the alphabet A*, is a function $f : [1,n] \to A \cup \{\lambda_1, ..., \lambda_m\}$ satisfying (a) $\forall i, f^{-1}(\lambda_i) \neq \emptyset$ (each λ_i occurs at least once), and (b) $\forall i < j,\ \min f^{-1}(\lambda_i) < \min f^{-1}(\lambda_j)$ (the first occurrences of the $\lambda's$ must be in order).

Denote the set of all m-parameter words of length n over A by $[A]\binom{n}{m}$. For $n, m, k \in \mathbb{Z}^+$, and two parameter words $f \in [A]\binom{n}{m}$ and $g \in [A]\binom{m}{k}$, define the *composition* $f \circ g \in [A]\binom{n}{k}$ as

$$(f \circ g)(i) = \begin{cases} f(i) & \text{if } f(i) \in A, \\ g(j) & \text{if } f(i) = \lambda_j. \end{cases}$$

For $f \in [A]\binom{n}{m}$, define the *space of f* as

$$sp(f) = \left\{ f \circ g : g \in [A]\binom{m}{0} \right\} \subseteq A^n.$$

An *m-dimensional combinatorial subspace (of A^n)* is the space of some m parameter word, and a 1-dimensional combinatorial subspace is called a *combinatorial line*.

For example, let $A = \{a, b\}$ and, dropping the parentheses and commas, $f = ab\lambda_1 a\lambda_2\lambda_1 \in [A]\binom{6}{2}$, and $g = b\lambda_1 \in [A]\binom{2}{1}$. Then $f \circ g$ is the function $abba\lambda_1 b \in [A]\binom{6}{1}$, and $sp(f \circ g) = \{abbaab, abbabb\}$.

5.6.2 The Hales-Jewett and Graham-Rothschild theorems

The Hales-Jewett theorem is a Ramsey-type theorem about parameter words.

Theorem 5.6.1 (Hales and Jewett, 1963 [71])**.** *Let $m, r \in \mathbb{Z}^+$, and let A be a finite alphabet. Then there exists a smallest $n = HJ(|A|, m, r) \in \mathbb{Z}^+$ such that for all $\Delta : [A]\binom{n}{0} \to \{1, \ldots, r\}$, there exists an $f \in [A]\binom{n}{m}$ such that Δ is constant on $sp(f)$.*

The proof is omitted, and can be found *e.g.*, [65], [79], or [121].

The following theorem, known as the Graham-Rothschild theorem, is a generalization of the Hales-Jewett theorem, where k-sets of A^n are being coloured, as opposed to just points in A^n (the proof is also omitted).

Theorem 5.6.2 (Graham and Rothschild, 1971 [64])**.** *Let A be a finite alphabet, and let k, m and r be positive integers. Then there exists a smallest positive integer $n = GR(|A|, k, m, r)$ such that for every colouring $\Delta : [A]\binom{n}{k} \to \{1, \ldots, r\}$ there* *exists a monochromatic* $f \in [A]\binom{n}{m}$*, i.e.,*

$$\Delta(f \circ g) = \Delta(f \circ h) \text{ for all } g, h \in [A]\binom{m}{k}.$$

The proof of the Graham-Rothschild theorem is also omitted.

5.7 Colouring complete subgraphs of larger order

Two proofs are given of the fact that for any $k \in \mathbb{Z}^+$ and for any graph G, there exists a graph F such that $F \xrightarrow{\text{ind}} (G)_2^{K_k}$. The first such proof uses the Graham-Rothschild theorem (Theorem 5.6.2), and the second using a theorem known as the Ordered Hypergraph theorem, which is described in Section 5.8.

5.7.1 Preliminary lemmas

Lemma 5.7.1 (see, *e.g.*, [112])**.** *For every finite graph G, there exists a set X such that G is an induced subgraph of the graph*

$$H(X) = (\mathcal{P}(X), \{\{A, B\} : A, B \neq \emptyset, A \cap B = \emptyset\}).$$

Proof. It suffices to show that for $X = V(G) \cup E(\overline{G})$, $G \preceq H(X)$. For each $v \in V(G)$, let $A_v = \{v\} \cup \{e \in E(\overline{G}) : v \in e\}$. Then for all $u, v \in V(G)$, $A_u \cap A_v = \emptyset$

iff $u \neq v$ and there exists $e = \{u, v\} \in E(G)$. Then the induced subgraph of $H(X)$ on the vertex set $\{A_v : v \in V(G)\}$ is an induced copy of G. $\square$

Observation 5.7.2. *For $X \subseteq Y$, $H(X) \preceq H(Y)$.*

Proof. Let $X \subseteq Y$. Then $\mathcal{P}(X) \subseteq \mathcal{P}(Y)$, and so the induced subgraph of $H(Y)$ on the vertices $\mathcal{P}(X)$ is a copy of $H(X)$. $\square$

5.7.2 First proof of induced Ramsey theorem

The main result can now be stated and proved.

Theorem 5.7.3 (see, *e.g.*, [102])**.** *Let G be a finite graph and let $k, r \in \mathbb{Z}^+$. Then there exists a graph F such that $F \xrightarrow{ind} (G)_r^{K_k}$.*

First proof of Theorem 5.7.3. (as given in [68]) By Lemma 5.7.1, any finite graph can be looked at as an induced subgraph of $H(X)$ for some set X. Thus, it suffices to fix some set X, and prove the theorem for $G = H(X)$.

Observation. Given a set $X = \{x_0, ..., x_{m-1}\}$, every K_k-subgraph of $H(X)$ corresponds to a set of k disjoint nonempty subsets $\{Y_0, Y_1, ..., Y_{k-1}\}$ of X. For an alphabet A of one element, say 0, $\{Y_0, \ldots, Y_{k-1}\}$ can be expressed as a single k-parameter word $f \in [A]\binom{m}{k}$, by mapping

$$f(i) = \lambda_j \text{ iff } x_i \in Y_j$$

and $f(i) = 0$ otherwise. Thus the K_k subgraphs of $H(X)$ are equivalent to the elements of $[1]\binom{m}{k}$.

Let $X = \{x_0, x_1, ..., x_{m-1}\}$, $G = H(X)$ be given. Let $A = \{0\}$, and fix $r, k \in \mathbb{Z}^+$. Let $n = GR(|A|, k, m, r)$ and let Y be a set of n elements. the claim is then that $H(Y) \xrightarrow{\text{ind}} (G)_r^{K_k}$.

Let $\Delta_0 : \binom{H(Y)}{K_k}_{\text{ind}} \to \{1, \ldots, r\}$. By the observation, Δ_0 is equivalent to a colouring $\Delta_0' : [A]\binom{n}{k} \to \{1, \ldots, r\}$. By the choice of n, there exists $f \in [A]\binom{n}{m}$ such that f is monochromatic under Δ_0'. Let

$$\mathcal{Z} = \{f^{-1}(\lambda_i) : i \in [0, m-1]\}$$

and let

$$FU(\mathcal{Z}) = \left\{ \bigcup_{i \in I} f^{-1}(\lambda_i) : I \subseteq [0, m-1] \right\}.$$

Since for all $i \neq j$, $f^{-1}(\lambda_i) \cap f^{-1}(\lambda_j) = \emptyset$, the bijection between $\mathcal{Z}$ and X that is given by $\phi(f^{-1}(\lambda_i)) = x_i$ also produces an isomorphism between $FU(\mathcal{X})$ and $\mathcal{P}(X)$. Therefore the induced subgraph of $H(Y)$ on the vertices $FU(\mathcal{X})$ is a copy of $G = H(X)$. Further, by the definition of f, the induced subgraph of $H(Y)$ on the vertices $FU(\mathcal{X})$ has all its K_k-subgraphs the same colour. $\square$

5.8 The Ordered Hypergraph Theorem

5.8.1 Definitions and statement

An *ordered hypergraph* $(H, \leq)$ is a hypergraph H, together with a total (linear) ordering $\leq$ on $V(H)$. Two ordered hypergraphs $(G, \leq_1)$ and $(H, \leq_2)$ are *isomorphic* if there exists an isomorphism f between G and H that is also order preserving (*i.e.*, for all $x, y \in V(G)$, $x \leq_1 y$ iff $f(x) \leq_2 f(y)$). An ordered hypergraph $(G, \leq_1)$ is an *ordered induced subhypergraph* of $(H, \leq_2)$, denoted by $(G, \leq_1) \preceq (H, \leq_2)$, if both $G \preceq H$ and for every $x, y \in V(G)$, $x \leq_1 y$ iff $x \leq_2 y$. The set of ordered induced subhypergraphs of $(G, \leq_1)$ isomorphic to $(H, \leq_2)$ is denoted $\binom{G,\leq_1}{H,\leq_2}_{\text{ind}}$. Notation may be abused slightly by writing $(G, \leq)$ and $(H, \leq)$, when what is meant is $(G, \leq_1)$ and $(H, \leq_2)$.

For $r \in \mathbb{Z}^+$, and ordered hypergraphs $(F, \leq)$, $(G, \leq)$, and $(H, \leq)$, write

$$(F, \leq) \xrightarrow{\text{ind}} (G, \leq)_r^{(H,\leq)}$$

if for every $\Delta : \binom{F,\leq}{H,\leq}_{\text{ind}} \to [1, r]$, there exists $(G', \leq) \in \binom{F,\leq}{G,\leq}_{\text{ind}}$ such that $\binom{G',\leq}{H,\leq}_{\text{ind}}$ is monochromatic under Δ.

The following theorem, known as the Ordered Hypergraph theorem, was published by Nešetřil and Rödl in 1977 [106] and Abramson and Harrington in 1978 [1].

Theorem 5.8.1 (The Ordered Hypergraph theorem)**.** *For every $r \in \mathbb{Z}^+$, and any ordered hypergraphs $(G, \leq)$ and $(H, \leq)$, there exists an ordered hypergraph $(F, \leq)$ such that*

$$(F, \leq) \xrightarrow{ind} (G, \leq)_r^{(H,\leq)}.$$

The proof of Theorem 5.8.1 is omitted, though one proof of it follows the same reasoning as the amalgamation proof that for any graph G, $R_{\mathrm{ind}}(G; K_2; 2)$ exists (Theorem 5.5.4). The interested reader can find proofs of the Ordered Hypergraph theorem in, *e.g.*, [1, 68, 106, 121]. Note that the Ordered Hypergraph theorem as presented here is a special case of Nešetřil and Rödl's version. The following is a useful consequence of the Ordered Hypergraph Theorem:

Corollary 5.8.2. *For any $k, r \in \mathbb{Z}^+$, and any ordered k-uniform hypergraphs $(G, \leq)$ and $(H, \leq)$, there exists an ordered k-uniform hypergraph $(F, \leq)$ such that*

$$(F, \leq) \xrightarrow{ind} (G, \leq)_r^{(H,\leq)}.$$

Proof. Let $(G, \leq)$ and $(H, \leq)$ be ordered k-uniform hypergraphs. By the Ordered Hypergraph theorem, there exists an ordered hypergraph $(F, \leq)$ such that

$$(F, \leq) \xrightarrow{\mathrm{ind}} (G, \leq)_r^{(H,\leq)}.$$

Let F' be the k-uniform hypergraph formed by removing all edges not containing exactly k elements from F. Then $(F', \leq) \xrightarrow{\mathrm{ind}} (G, \leq)_r^{(H,\leq)}$, and the proof is complete.

□

5.8.2 Second proof of the induced Ramsey theorem

Theorem 5.8.1 nearly immediately implies that for any $k \in \mathbb{Z}^+$, and any graph G, $R_{\text{ind}}(G; K_k; 2)$ exists (Theorem 5.7.3).

Second Proof of Theorem 5.7.3. (as given in [121]) Let G be any graph, let $k \in \mathbb{Z}^+$, and let $H = K_k$. Impose any total (linear) ordering on $V(G)$ and $V(H)$ to form the ordered graphs $(G, \leq)$ and $(H, \leq)$. By the corollary to the Ordered Hypergraph theorem (Corollary 5.8.2), there exists an ordered graph $(F, \leq)$ such that $(F, \leq) \xrightarrow{\text{ind}} (G, \leq)_r^{(H,\leq)}$. It suffices to prove that $F \xrightarrow{\text{ind}} (G)_r^H$. Let $\Delta : \binom{F}{H}_{\text{ind}} \to [1, r]$. Note that since H is complete, there is a one-to-one correspondence between $\binom{F}{H}_{\text{ind}}$ and $\binom{F,\leq}{H,\leq}_{\text{ind}}$. Thus Δ induces a colouring $\Delta' : \binom{F,\leq}{H,\leq}_{\text{ind}} \to [1, r]$ by $\Delta'(H', \leq) = \Delta(H')$. By the choice of $(F, \leq)$, there exists a copy $(G', \leq) \in \binom{F,\leq}{G,\leq}_{\text{ind}}$ monochromatic under Δ'. The graph $G' \in \binom{F}{G}_{\text{ind}}$ is then monochromatic under Δ. □

5.8.3 Application: The Ordering Property

The ordering property for a graph, introduced by Nešetřil and Rödl in 1975 [103], is a consequence of the Ordered Hypergraph theorem. While interesting in its own sake, it can also be applied to prove some Ramsey theorems about unordered graphs. One such application is found in Chapter 10 (Theorem 10.1.2).

Let $(H, \leq)$ be an ordered graph, and let G be an unordered graph. The graph G is said to *satisfy the ordering property for* $(H, \leq)$, denoted

$$G \xrightarrow{\text{ind}}_{\text{ord}} (H, \leq),$$

iff for every possible ordering $\leq^*$ of $V(G)$, there exists $(H', \leq) \in \binom{G, \leq^*}{H, \leq}_{\text{ind}}$.

Theorem 5.8.3 (Nešetřil and Rödl, 1975 [103])**.** *For every ordered graph* $(H, \leq)$*, there exists an (unordered) graph* G *so that* $G \xrightarrow{ind}_{ord} (H, \leq)$*.*

Proof. Let $(H, \leq)$ be an ordered graph. If H is not connected, add a new vertex to $(H, \leq)$, and connect it to each component of H (the extra vertex can be removed later without a problem). Let $v_1 < v_2 < ... < v_m$ be the vertices of $(H, \leq)$. Let $(H, \leq^{-1})$ be the ordered graph where for all $v_i, v_j \in V(H)$, $v_i \leq^{-1} v_j$ iff $v_j \leq v_i$.

Let $(H^*, \leq^*)$ be the graph made up of the ordered disjoint union of $(H, \leq)$ and $(H, \leq^{-1})$ where all vertices in $(H, \leq^{-1})$ come after those in $(H, \leq)$; that is, $H^* = H \,\dot{\cup}\, H$ and for v_i in $(H, \leq)$, and v_j in $(H, \leq^{-1})$, $v_i \leq^* v_j$ (see Figure 5.10).

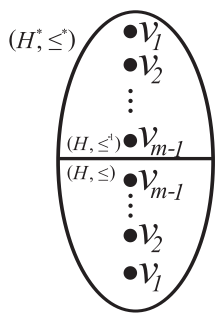


Figure 5.10: The graph $(H^*, \leq^*)$

By the corollary to the Ordered Hypergraph theorem (Corollary 5.8.2), given the ordered graphs $(H^*, \leq^*)$ and $(K_2, \leq)$ (note K_2 has only one ordering up to isomorphism), there exists an ordered graph $(G, \leq')$ such that $(G, \leq') \xrightarrow{\text{ind}} (H^*, \leq^*)_2^{K_2}$.

Claim. The graph G satisfies $G \xrightarrow{\text{ind}}_{\text{ord}} (H, \leq)$.

Let $\leq''$ be an arbitrary ordering on $V(G)$. Then it remains to show that $(H, \leq) \preceq (G, \leq'')$. Let Δ be a 2-colouring of the edges of $(G, \leq')$, defined as follows: for $x \leq' y$,

$$\Delta(\{x, y\}) = \begin{cases} \text{red if } x \leq'' y \\ \\ \text{blue if } y \leq'' x \end{cases}$$

Then, by the choice of $(G, \leq')$, there exists a monochromatic $(H_0^*, \leq^*) \in \binom{G, \leq'}{H^*, \leq^*}_{\text{ind}}$. If $(H_0^*, \leq^*)$ is monochromatic red, then the non-inverted copy of $(H, \leq)$ in $(H_0^*, \leq^*)$ is a copy of $(H, \leq)$ in $(G, \leq'')$, and the theorem is proved. If not, then $(H_0^*, \leq^*)$ is monochromatic blue, and the inverted copy of $(H, \leq)$ in $(H_0^*, \leq^*)$ is a copy of $(H, \leq)$ in $(G, \leq'')$.

Note that it was required that H is connected, since if it weren't, $(G, \leq'')$ could have the components of $(H, \leq)$ and/or $(H, \leq^{-1})$ backwards, and a copy of $(H, \leq)$ ordered exactly as needed could not be guaranteed. Finally, if a new vertex v was added in the beginning to make H connected, then v and the edges v is incident with can now be removed to yield the desired induced subgraph. $\square$

5.8.4 The Partite lemma

Nešetřil and Rödl's proof of the Ordered Hypergraph theorem used a generalization of the Bipartite Lemma (Theorem 5.5.3), generalized to more complex set systems. In fact, according to Nešetřil [102], the amalgamation generally follows the same line for any type of partite systems: (1) definition of the systems and the amalgamation, (2) a partite lemma, and (3) a partite construction. A generalization of the Bipartite Lemma to ordered hypergraphs is presented here.

Let $(H, \leq)$ be an ordered hypergraph. Then for two disjoint sets $X, Y \subseteq V(H)$, write $X < Y$ if for all $x \in X$ and all $y \in Y$, $x < y$. For $a \in \mathbb{Z}^+$, $(H, \leq)$ is *a-partite* iff there exists a partition $V(H) = V_1 \cup \cdots \cup V_a$ such that $V_1 < \cdots < V_a$, and for any edge $e \in E(H)$, there exists $i, j \in [1, a]$, $i \neq j$, such that $e \cap V_i \neq \emptyset$ and $e \cap V_j \neq \emptyset$. An a-partite ordered hypergraph $(H, \leq)$ with partition $V(H) = V_1 \cup \cdots \cup V_a$ is *transversal* if for all $i \in [1, a]$, $|V_i| = 1$.

Let $a \in \mathbb{Z}^+$. Two a-partite ordered hypergraphs $(G, \leq)$ and $(H, \leq)$ with partitions $V(G) = V_1 \cup \cdots \cup V_a$ and $V(H) = X_1 \cup \cdots \cup X_a$ are said to be *partite isomorphic* if there exists an isomorphism f between G and H that is order preserving, and has the property that for all $i \in [1, a]$, $f(V_i) = X_i$. The a-partite ordered hypergraph $(G, \leq_1)$ is said to be a *partite ordered induced subhypergraph* of $(H, \leq_2)$ if $G \preceq H$, for every $x, y \in V(G)$, $x \leq_1 y$ iff $x \leq_2 y$, and if for all $i \in [1, a]$, $V_i \subseteq X_i$. The set of *partite ordered induced subhypergraphs of* $(G, \leq_1)$ *isomorphic to* $(H, \leq_2)$ is denoted $\binom{G,\leq_1}{H,\leq_2}_{\text{part}}$.

For $a, r \in \mathbb{Z}^+$, and a-partite ordered hypergraphs $(F, \leq)$, $(G, \leq)$, and $(H, \leq)$, write

$$(F, \leq) \xrightarrow{\text{ind}}_{\text{part}} (G, \leq)_r^{(H,\leq)}$$

if for every $\Delta : \binom{F,\leq}{H,\leq}_{\text{part}} \to [1, r]$, there exists $(G', \leq) \in \binom{F,\leq}{G,\leq}_{\text{part}}$ such that $\binom{G',\leq}{H,\leq}_{\text{part}}$ is monochromatic under Δ.

First, a specific case of the partite lemma (when H is complete) is proved.

Theorem 5.8.4 (Nešetřil and Rödl, 1989 [114])**.** *Let* $r, a \in \mathbb{Z}^+$*, and let* $(G, \leq)$ *and* $(H, \leq)$ *be a-partite ordered hypergraphs. Moreover, let* H *be transversal in* G *and complete. Then there exists an a-partite ordered hypergraph* $(F, \leq)$ *such that*

$$(F, \leq) \xrightarrow{ind}_{part} (G, \leq)_r^{(H,\leq)}.$$

Proof. (as given in [68]) Let $H = (Y_1 \cup \cdots \cup Y_a, \mathcal{H})$, $G = (X_1 \cup \cdots \cup X_a, \mathcal{G})$, H transversal and complete. Let $Y = \bigcup_{i=1}^a Y_i$ and $X = \bigcup_{i=1}^a X_i$.

First note that since H is both complete and transversal, for any a-partite hypergraph F, there is a one-to-one correspondence between $\binom{F}{H}_{\text{part}}$ and $\binom{F,\leq}{H,\leq}_{\text{part}}$.

If there are vertices of G not contained in some copy of H in $\binom{G}{H}_{\text{part}}$, and if for $G^* = G[\{v \in H' : H' \in \binom{G}{H}_{\text{part}}\}]$ an ordered hypergraph $(F^*, \leq)$ is found such that $(F^*, \leq) \xrightarrow{\text{ind}}_{\text{part}} (G^*, \leq)_r^{(H,\leq)}$, then each copy of G^* in F^* can be independently enlarged to a copy of G to form a graph F such that $(F, \leq) \xrightarrow{\text{ind}}_{\text{part}} (G, \leq)_r^{(H,\leq)}$). Therefore, without loss of generality, assume that each vertex of G is contained in some copy of H in $\binom{G}{H}_{\text{part}}$.

Let $t = \left|\binom{G}{H}_{\text{part}}\right|$ and let $N = HJ(t, 1, r)$, the Hales-Jewett number (defined in Theorem 5.6.1). Let $\mathcal{G}'$ be the set of edges in $\mathcal{G}$ that are contained in some partite copy of $(H, \leq)$ in $\binom{G}{H}_{\text{part}}$ (that is, $\mathcal{G}' = \{e \in E(H') : H' \in \binom{G}{H}_{\text{part}}\}$), and let $\mathcal{G}'' = \mathcal{G} \setminus \mathcal{G}'$.

Recall that for any S, and integer k, S^k denotes the direct product of S with itself k times. For all $i \in [1, a]$, define $Z_i = X_i^N$. That is, each element $\mathbf{v} \in Z_i$ is of the form

$$\mathbf{v} = (v_1, \ldots, v_N) : \forall j \in [1, N],\ v_j \in X_i.$$

Let $Z = \bigcup_{i=1}^{a} Z_i$. For all $j \in [1, N]$, define a projection $\pi_j : Z \to X$ defined by

$$\pi_j : (v_1, \ldots, v_N) \mapsto v_j.$$

Then for all $j \in [1, N]$, and for all $i \in [1, a]$, $\pi_j(Z_i) = X_i$ (π_j preserves parts), and π_j is onto. For any $\Gamma \subseteq Z$, let $\pi_j(\Gamma) = \{\pi_j(\mathbf{v}) : \mathbf{v} \in \Gamma\}$.

Define the edge set $\mathcal{F}$ as follows: for $\Gamma \subseteq Z$, $\Gamma \in \mathcal{F}$ iff one of the following condition is satisfied: (a) for all $j \in [1, N]$, $\pi_j(\Gamma) \in \mathcal{G}'$, or (b) there exists $J \subseteq [1, N]$ and $B \in \mathcal{G}''$ such that for all $j \in J$, $\pi_j(\Gamma) = B$, and for all $j \notin J$, $\pi_j(\Gamma) \in \mathcal{G}'$.

Notice that if every edge of G is a member of some partite copy of H, then $\mathcal{G}''$ is empty, and (b) never occurs. Finally, let $F = (Z_1 \cup \cdots \cup Z_a, \mathcal{F})$, and order the vertices of F lexicographically with respect to the ordering of $V(G)$. It suffices to prove that $(F, \leq) \xrightarrow{\text{ind}}_{\text{part}} (G, \leq)_r^{(H,\leq)}$.

Let $\Delta : \binom{F}{H}_{\text{part}} \to [1, r]$. Suppose $H' \in \binom{F}{H}_{\text{part}}$ and $Y' = V(H')$. Then for every $e \subseteq Y'$, it follows that e is an edge in F (since H is complete and transversal) and therefore for every $j \in [1, N]$, $\pi_j(e)$ is an edge in G (by the definition of the edges in F). Therefore for every $j \in [1, N]$, $\pi_j(Y')$ induces a (partite) copy of H in G.

Similarly, if $Y^* \in [Z]^a$ is such that for every $j \in [1, N]$, $\pi_j(Y^*)$ induces a (partite) copy of H in G, then for every $e \subseteq \pi_j(Y^*)$, $e \in \mathcal{G}'$, and therefore by (a), Y^* induces a (partite) copy of H in F.

Recall that for any set A, and any $n, m \in \mathbb{Z}^+$, $[A]\binom{n}{m}$ denotes the set of m-parameter words of length n over the alphabet A (see Section 5.6).

For $Y' \in [Z]^a$, the subgraph of F induced by Y' satisfies

$$\begin{aligned} F[Y'] \in \binom{F}{H}_{\text{part}} \quad &\text{iff} \quad \forall j \in [1, N], (G[\pi_j(Y')], \leq) \in \binom{G}{H}_{\text{part}} \\ &\text{iff} \quad (G[\pi_1(Y')], \leq), \cdots (G[\pi_N(Y')], \leq) \in \left[\binom{G}{H}_{\text{part}}\right]\binom{N}{0}. \end{aligned}$$

Enumerate $\binom{G}{H}_{\text{part}} = \{H_1, \ldots, H_t\}$.

Define $\Delta' : \left[\binom{G}{H}_{\text{part}}\right]\binom{N}{0} \to [1, r]$ by (for $H' \in \binom{F}{H}_{\text{part}}$)

$$\Delta'(\ (G[\pi_1(V(H'))], \leq), (G[\pi_2(V(H'))], \leq), \ldots, (G[\pi_N(V(H'))], \leq)\ ) = \Delta(H').$$

By the choice if N, there exists $h \in \left[\binom{G}{H}_{\text{part}}\right]\binom{N}{1}$ such that Δ' is constant on $sp(h)$.

For each $f \in \left[\binom{G}{H}_{\text{part}}\right]\binom{N}{0}$, define

$$\phi(f) = \left\{ \mathbf{v} \in Z : \forall j \in [1, N], \pi_j(\mathbf{v}) \in f(j) \in \binom{G}{H}_{\text{part}} \right\}.$$

For $g \in \left[\binom{G,\leq}{H,\leq}_{\text{part}}\right]\binom{N}{1}$, let $\phi(g) = \bigcup_{f \in sp(g)} \phi(f)$.

To finish proving Theorem 5.8.4, it suffices to prove the following claim:

Claim. The set $\phi(h) \subseteq Z$ induces a monochromatic copy of $(G, \leq)$ (under Δ) in $(F, \leq)$.

Proof of Claim. Define the function $\Psi : X \to \phi(h)$ as follows: fix $v \in X$ and assume that for some $k \in [1, a]$, $v \in X_k$. For each $j \in [1, N]$, define

$$v_j = \begin{cases} v & \text{if } h(j) = \lambda, \text{ or} \\ \text{the vertex of } h(j) \text{ in } X_k & \text{if } h(j) \neq \lambda. \end{cases}$$

(note that this definition is well defined since H is transversal). Then let $\Psi(v) = (v_1, \ldots, v_N)$. The function Ψ is then an order preserving isomorphism between $(G, \leq)$ and an induced subgraph of $(F, \leq)$, with each copy of $(H, \leq)$ inside the same colour. This proves the claim, and the theorem. $\square$

The more general version of the partite lemma can now be stated and proved. The proof of Theorem 5.8.5 presented below is as given in [68, p. 65].

Theorem 5.8.5 (The Partite Lemma, Nešetřil and Rödl, 1989 [114])**.** *Fix $r, a \in \mathbb{Z}^+$, and let $(G, \leq)$ and $(H, \leq)$ be a-partite ordered hypergraphs. Moreover, let H be transversal in G, but not necessarily complete. Then there exists an a-partite ordered hypergraph $(F, \leq)$ such that*

$$F \xrightarrow{ind}_{part} (G, \leq)_r^{(H, \leq)}.$$

Proof. If H is complete, the corollary is exactly Theorem 5.8.5. If H is not complete, let H' be the complete (transversal) a-partite hypergraph on $V(H)$ (call the extra edges in H' 'dummy' edges). Let $G = (X_1 \cup \cdots \cup X_a, \mathcal{E}_1)$, and let $X = \bigcup_{i=1}^{a} X_i$. Define a set $\mathcal{E}_2 \subseteq \mathcal{P}(X)$ as follows: for any $e \in \mathcal{P}(X)$ such that $\forall i \in [1, a]$ $|e \cap X_i| \leq 1$,

$$e \in \mathcal{E}_2 \text{ iff } \begin{cases} e \in E(G) \text{ and } \exists e' \in E(H) \text{ s.t. } \forall i \in [1, a] V_i \cap e' \text{ iff } V_i \cap e \\ e \notin E(G) \text{ and } \nexists e' \in E(H) \text{ s.t. } \forall i \in [1, a] V_i \cap e' \text{ iff } V_i \cap e. \end{cases}$$

Define $G' = (X_1 \cup \cdots \cup X_a, \mathcal{E}_2)$ (see Figure 5.11 for an example of the construction of H' and G'). By Theorem 5.8.5, there exists an ordered hypergraph $(F', \leq)$ such that $(F', \leq) \xrightarrow{\text{ind}}_{\text{part}} (G', \leq)_r^{(H', \leq)}$. By the construction of $\mathcal{E}_2$, there is a one-to-one correspondence between the $(H', \leq)$ partite subgraphs of $(G', \leq)$ and the $(H, \leq)$ partite subgraphs of $(G, \leq)$. Therefore, it is well defined to remove the 'dummy'

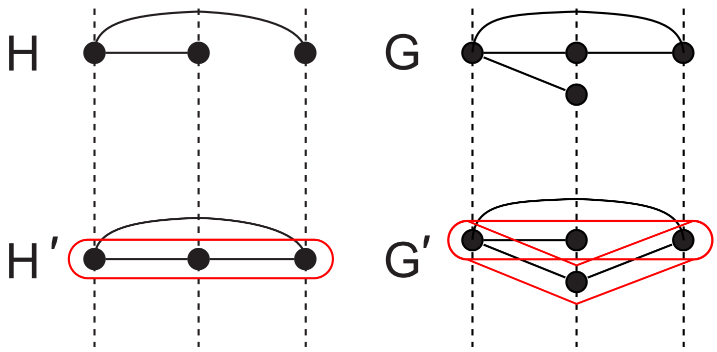


Figure 5.11: Example H, H', G, and G'

edges from each copy of H' in F'. Let F be the hypergraph formed by removing the 'dummy' edges from each copy of H' in F'. Then $(F, \leq) \xrightarrow{\text{ind}}_{\text{part}} (G, \leq)_r^{(H,\leq)}$. $\square$

Chapter 6

Extremal graph theory

In proving results in graph Ramsey theory, other results known as "density results" are sometimes useful. Density results involve the existence of sufficiently many "objects" to force some "property". For example, a traditional density theorem for graphs is Mantel's theorem: every graph on n vertices with at least $\left\lfloor \frac{n^2}{4} \right\rfloor + 1$ edges contains a triangle.

6.1 Extremal Numbers

Given $n \in \mathbb{Z}^+$, and any graph G, the *extremal number for n and G*, denoted $ex(n, G)$, is the maximum number of edges a graph on n vertices can contain without containing a subgraph isomorphic to G. For example, if $V(G) > n$ then $ex(n, G) = \binom{n}{2}$. The extremal numbers have been the subject of a lot of research (see *e.g.* [9]).

Density results often yield related Ramsey results. For example, consider the following proposition.

Proposition 6.1.1. *If n and r are positive integers, and G is a graph such that $\frac{1}{r}\binom{n}{2} > ex(n, G)$, then $R(G; r) \leq n$.*

Proof. Let $n, r \in \mathbb{Z}^+$ be such that $\frac{1}{r}\binom{n}{2} > ex(n, G)$, and let $\Delta : E(K_n) \to [1, r]$. Since the average number of edges in any colour class is $\frac{1}{r}\binom{n}{2} > ex(n, G)$, there exists at least one colour class, say in colour c, with more than $ex(n, G)$ edges. This colour class then has more than $ex(n, G)$ edges, and therefore there is a copy of G monochromatic in the c-th colour. $\square$

Proposition 6.1.1 shows that as soon as one has an upper bound on the extremal numbers, there is an associated Ramsey result.

6.2 Turán's Theorem

By Mantel's theorem (mentioned above), $ex(n, K_3) \leq \frac{n^2}{4}$. As well, the graph $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ contains no triangles and, if n is even, has $\left\lfloor \frac{n^2}{4} \right\rfloor$ edges, which shows that $ex(n, K_3) \geq \left\lfloor \frac{n^2}{4} \right\rfloor$. It then follows that $ex(n, K_3) = \left\lfloor \frac{n^2}{4} \right\rfloor$. For all $n, k \in \mathbb{Z}^+$, Turán's Theorem (below) produces the value of, $ex(n, K_k)$.

Let $n, k \in \mathbb{Z}^+$. The *k-partite Turán graph on n vertices*, denoted $T(n, k)$, is the complete k-partite graph with partite sets $V_1, \ldots, V_k$, and on n vertices such that the partite sets are as close as possible in size; that is, for all $i, j \in [1, k]$, $||V_i| - |V_k|| \leq 1$. Let $q, r \in \mathbb{Z}^+$ be such that $r \in [0, k-1]$, and $n = qk + r$. Then r of the partite sets have $q + 1$ vertices, and the remaining have q vertices (see Figure 6.1). Note that there is no copy of K_{k+1} in $T(n, k)$. Let $t(n, k) = |E(T(n, k))|$.

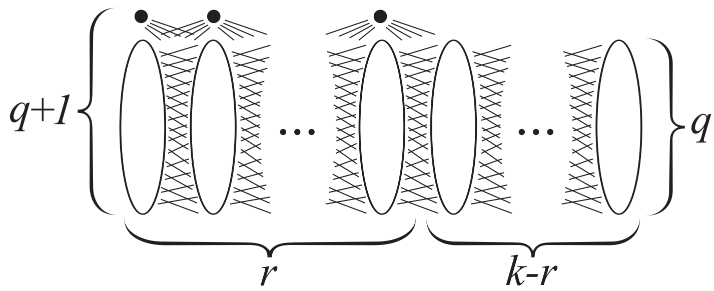


Figure 6.1: The Turán graph $T(n, k)$

Theorem 6.2.1 (Turán's theorem, 1941/1954 [140, 141])**.** *Let G be a graph on n vertices. If $|E(G)| > t(n, k)$, then there is a subgraph of G isomorphic to K_{k+1}.*

Furthermore, $T(n,k)$ is the unique (up to isomorphism) K_{k+1}-free graph with n vertices and $t(n,k)$ edges.

The proof of Turán's theorem is omitted. There are a number of known proofs of Turán's theorem (see, *e.g.*, [3, 7, 79]).

Lemma 6.2.2. *For $n, k \in \mathbb{Z}^+$, and $q, r \in \mathbb{Z}^+$ such that $r \in [0, k-1]$ and $n = qk + r$,*

$$t(n,k) = \frac{1}{2}(k)q(k-1)q + \binom{r}{2} + r(k-1)q.$$

Proof. The first term $\frac{1}{2}(k)q(k-1)q = \binom{k}{2}q^2$ counts the edges fully within the bottom section of the graph. The second term $\binom{r}{2}$ counts edges just in the top row. The third term $r(k-1)q$ counts edges connecting the top row with the remainder of the graph. □

The following proposition is used in Chapter 7.

Proposition 6.2.3. *For all $n, k \in \mathbb{Z}^+$,*

$$t(n, k-1) \leq \frac{1}{2}n^2\left(\frac{k-2}{k-1}\right).$$

Proof. Let $q, r \in \mathbb{Z}^+$, $r \in [0, k-2]$, be such that $n = q(k-1) + r$. Then

$$\begin{aligned} t(n,k-1) &= \frac{1}{2}(k-1)q(k-2)q + \binom{r}{2} + r(k-2)q && \text{(by Lemma 6.2.2)} \\ &= \frac{1}{2}(k-1)\left(\frac{n-r}{k-1}\right)(k-2)\left(\frac{n-r}{k-1}\right) + \binom{r}{2} + r(k-2)\left(\frac{n-r}{k-1}\right) \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2}(n-r)^2\left(\frac{k-2}{k-1}\right)+\frac{r}{2}\left(2(n-r)\left(\frac{k-2}{k-1}\right)+r-1\right) \\
&= \left(\frac{1}{2}n^2+\frac{-2nr+r^2}{2}\right)\left(\frac{k-2}{k-1}\right)+\frac{r}{2}\left(2(n-r)\left(\frac{k-2}{k-1}\right)+r-1\right) \\
&= \frac{1}{2}n^2\left(\frac{k-2}{k-1}\right)+\frac{r}{2}\left(-r\left(\frac{k-2}{k-1}\right)+r-1\right) \\
&= \frac{1}{2}n^2\left(\frac{k-2}{k-1}\right)+\frac{r}{2}\left(r\left(1-\frac{k-2}{k-1}\right)-1\right) \\
&= \frac{1}{2}n^2\left(\frac{k-2}{k-1}\right)+\frac{r}{2}\left(r\left(\frac{1}{k-1}\right)-1\right) \\
&\leq \frac{1}{2}n^2\left(\frac{k-2}{k-1}\right)+\frac{r}{2}\left(\frac{k-2}{k-1}-1\right) \qquad \text{(since } r \leq k-2\text{)} \\
&\leq \frac{1}{2}n^2\left(\frac{k-2}{k-1}\right) \qquad \left(\text{since } \frac{k-2}{k-1} < 1\right). \qquad \square
\end{aligned}$$

6.3 The symmetric hypergraph theorem

The theorem known as the Symmetric Hypergraph theorem seems to have been first published by Graham, Rothschild, and Spencer [65, pp. 99–103]. The authors also use the Symmetric Hypergraph theorem to produce a relationship between the graph Ramsey numbers $R(G; K_2; r)$ and the extremal numbers $ex(n, G)$. According to the authors [65, p. 109], both the theorem and this application have been part of the "folk literature" [65, p. 110] for some time.

All theorems in this section, unless otherwise stated, are found in [65].

6.3.1 Preliminaries

For a hypergraph H, an *automorphism of* H is an isomorphism between H and itself. The set of all automorphisms on H forms a group under composition, and is denoted $\mathrm{Aut}(H)$. A group G acting on a set X is *transitive* iff for any elements $u, v \in X$, there exists a $\sigma \in G$ such that $\sigma(u) = v$. A hypergraph H is *symmetric* (or *vertex-symmetric*) if $\mathrm{Aut}(H)$ is transitive. The following lemma is used to prove the Symmetric Hypergraph theorem.

Lemma 6.3.1. *Let $H = (S, \mathcal{E})$ be a symmetric hypergraph, and let T and U be subsets of S. There exists $\sigma \in Aut(H)$ such that*

$$|\sigma(T) \cap U| \geq \frac{|T||U|}{|S|}.$$

Proof. Let $G = \mathrm{Aut}(H)$. Count the set of ordered triples

$$P = \{(\sigma, t, u) : \sigma \in G, t \in T, u \in U, \sigma(t) = u\}$$

in two ways.

To count the triples in P in the first way, for any $s \in S$, $t \in T$, define $X_{t,s} = \{\sigma \in G : \sigma(t) = s\}$. Let $t \in T$, $a, b \in S$. Since G is transitive, there exists $\phi \in G$ such that $\phi(a) = b$, and therefore,

$$\phi \circ X_{t,a} \quad = \quad \{\phi \circ \sigma : \sigma \in X_{t,a}\},$$

$$
\begin{aligned}
&= \{\phi \circ \sigma : \sigma \in G, \sigma(t) = a\}, \\
&\subseteq \{\sigma' \in G : \sigma'(t) = b\}, \\
&= X_{t,b}.
\end{aligned}
$$

Consider the function $f_t : X_{t,a} \to \phi \circ X_{t,a}$ where $f_t(\sigma) = \phi \circ \sigma$. Assume for some $\sigma_1, \sigma_2 \in G$, that $f_t(\sigma_1) = f_t(\sigma_2)$. Then,

$$
\begin{aligned}
f_t(\sigma_1) &= f_t(\sigma_2) \\
\text{iff} \quad \phi \circ \sigma_1 &= \phi \circ \sigma_2, \\
\text{iff} \quad \phi^{-1} \circ \phi \circ \sigma_1 &= \phi^{-1} \circ \phi \circ \sigma_2, \qquad (\phi^{-1} \text{ exists since } G \text{ is a group}) \\
\text{iff} \quad \sigma_1 &= \sigma_2.
\end{aligned}
$$

Therefore f_t is one-to-one, and thus $|X_{t,a}| \leq |X_{t,b}|$. By the same logic, replacing a and b, it follows that $|X_{t,b}| \leq |X_{t,a}|$. Therefore $|X_{t,a}| = |X_{t,b}|$. Since a, b were arbitrary elements of S, and since for any $t \in T$, $a, b \in S$, $X_{t,a} \cap X_{t,b} = \emptyset$, it follows that for any $s \in S$,

$$
|X_{t,s}| = \frac{|G|}{|S|}. \tag{6.1}
$$

Therefore,

$$
\begin{aligned}
|P| &= |\{(\sigma, t, u) : \sigma \in G, t \in T, u \in U, \sigma(t) = u\}| \\
&= \sum_{t \in T} \sum_{u \in U} |\{\sigma \in G : \sigma(t) = u\}|
\end{aligned}
$$

$$\begin{aligned}
&= \sum_{t\in T}\sum_{u\in U} |X_{t,u}| \\
&= \sum_{t\in T}\sum_{u\in U} \frac{|G|}{|S|} \qquad\qquad \text{(by equation (6.1))} \\
&= \frac{|T||U||G|}{|S|}. \qquad (6.2)
\end{aligned}$$

To count the triples in P in the second way, for each $\sigma \in G$, let

$$Y_\sigma = \{(t,u) : t \in T, u \in U, \sigma(t) = u\}.$$

Then,

$$\begin{aligned}
|P| &= |\{(\sigma,t,u) : \sigma \in G, t \in T, u \in U, \sigma(t) = u\}| \\
&= \sum_{\sigma\in G} |\{(t,u) : t \in T, u \in U, \sigma(t) = u\}| \\
&= \sum_{\sigma\in G} |Y_\sigma|.
\end{aligned}$$

The average cardinality of the Y_σ sets is

$$\frac{1}{|G|}\sum_{\sigma\in G} |Y_\sigma|.$$

So there must be at least one element in G, say σ_0, so that Y_{σ_0} has at least $\frac{1}{|G|}\sum_{\sigma\in G} |Y_\sigma|$ elements. Therefore, by equation (6.2),

$$|Y_{\sigma_0}| \geq \frac{1}{|G|}\sum_{\sigma\in G} |Y_\sigma| = \frac{|P|}{|G|} = \frac{|T||U|}{|S|}.$$

It now suffices to show that $|Y_{\sigma_0}| \leq |\sigma_0(T) \cap U|$. Let $\pi_2 : Y_{\sigma_0} \to \sigma_0(T) \cap U$ be the function projecting each pair (t,u) onto the second coordinate u, that is, $\pi(t,u) = u$.

Since σ_0 is one-to-one, the function π is also one-to-one, and thus $|Y_{\sigma_0}| \leq |\sigma_0(T) \cap U|$, and so,

$$\frac{|T||U|}{|S|} \leq |Y_{\sigma_0}| \leq |\sigma_0(T) \cap U|. \qquad \square$$

6.3.2 Statement and proof

Recall that for any graph G, $\alpha(G)$ is the independence number of G, that is, the maximum number of vertices containing no edge in G.

Theorem 6.3.2 (Symmetric Hypergraph theorem)**.** *Let $H = (S, \mathcal{E})$ be a symmetric hypergraph, $\mathcal{E} \neq \emptyset$, and let $m = |S| \geq 1$. Then,*

$$m\left(1 - \frac{\alpha(H)}{m}\right)^{\chi(H)-1} \geq 1,$$

and therefore

$$\chi(H) \leq 1 + \frac{\ln m}{-\ln\left(1 - \frac{\alpha(H)}{m}\right)}.$$

Proof. Let $T \subseteq S$ be an independent set such that $|T| = \alpha(H)$. Fix $r \in \mathbb{Z}^+$ such that

$$m\left(1 - \frac{\alpha(H)}{m}\right)^{r} < 1 \qquad \text{and} \qquad m\left(1 - \frac{\alpha(H)}{m}\right)^{r-1} \geq 1.$$

Inductively define $U_1, \ldots, U_r \subseteq S$ and functions $\sigma_1, \ldots, \sigma_r \in \mathrm{Aut}(H)$ as follows: let σ_1 be the identity automorphism on H, and define $U_1 = S \setminus T$.

For the inductive step in the definition, let $i \in [2, r]$, and assume that U_{i-1} and σ_{i-1} have been defined. By Lemma 6.3.1, there exists $\sigma_i \in \mathrm{Aut}(H)$ such that

$$|\sigma_i(T) \cap U_{i-1}| \geq \frac{|T||U_{i-1}|}{m} = \frac{\alpha(H)}{m}|U_{i-1}|.$$

Finally, define $U_i = U_{i-1} \setminus \sigma_i(T)$. This completes the inductive definition.

Note that for all $i \in [2, r]$,

$$|U_i| = |U_{i-1}| - |\sigma_i(T) \cap U_{i-1}| \leq |U_{i-1}| - \frac{\alpha(H)}{m}|U_{i-1}| = |U_{i-1}|\left(1 - \frac{\alpha(H)}{m}\right).$$

Therefore,

$$\begin{aligned}
|U_r| &\leq |U_{r-1}|\left(1 - \frac{\alpha(H)}{m}\right) \\
&\leq |U_{r-2}|\left(1 - \frac{\alpha(H)}{m}\right)^2 \\
&\vdots \\
&\leq |U_1|\left(1 - \frac{\alpha(H)}{m}\right)^{r-1} \\
&= (m - \alpha(H))\left(1 - \frac{\alpha(H)}{m}\right)^{r-1} \\
&= m\left(1 - \frac{\alpha(H)}{m}\right)^{r} \\
&< 1.
\end{aligned}$$

Thus $|U_r| < 1$, and so necessarily $U_r = \emptyset$. Hence,

$$S = \sigma_1(T) \cup \cdots \cup \sigma_r(T) \tag{6.3}$$

(not necessarily disjoint unions). Since each σ_i is an automorphism, and T is independent, each $\sigma_i(T)$ is an independent set. Let

$$\begin{aligned}
T_1 &= \sigma_1(T), \\
T_2 &= \sigma_2(T) \setminus T_1, \\
T_3 &= \sigma_3(T) \setminus (T_1 \cup T_2), \\
\vdots & \quad \vdots \\
T_r &= \sigma_r(T) \setminus (T_1 \cup \cdots \cup T_{r-1}).
\end{aligned}$$

Then by construction, for any $i, j \in [1, r]$, $i \neq j$, $T_i \cap T_j = \emptyset$, and by equation (6.3),

$$S = T_1 \cup T_2 \cup \cdots \cup T_r.$$

The T_i's form an r-colouring of S such that no edge is monochromatic, and therefore $\chi(H) \leq r$, from which it follows that

$$m\left(1 - \frac{\alpha(H)}{m}\right)^{\chi(H)-1} \geq m\left(1 - \frac{\alpha(H)}{m}\right)^{r-1} \geq 1.$$

The upper bound on $\chi(H)$ is proved by the following straightforward algebra:

$$\begin{aligned}
m\left(1 - \frac{\alpha(H)}{m}\right)^{\chi(H)-1} &\geq 1 \\
\ln\left[m\left(1 - \frac{\alpha(H)}{m}\right)^{\chi(H)-1}\right] &\geq 0 \\
\ln m + \ln\left(1 - \frac{\alpha(H)}{m}\right)^{\chi(H)-1} &\geq 0 \\
(\chi(H) - 1)\ln\left(1 - \frac{\alpha(H)}{m}\right) &\geq -\ln m
\end{aligned}$$

$$\begin{aligned}
\chi(H)\ln\left(1-\frac{\alpha(H)}{m}\right)-\ln\left(1-\frac{\alpha(H)}{m}\right) &\geq -\ln m \\
\chi(H)\ln\left(1-\frac{\alpha(H)}{m}\right) &\geq \ln\left(1-\frac{\alpha(H)}{m}\right)-\ln m \\
\chi(H) &\leq \frac{\ln\left(1-\frac{\alpha(H)}{m}\right)-\ln m}{\ln\left(1-\frac{\alpha(H)}{m}\right)} \\
\chi(H) &\leq 1+\frac{\ln m}{-\ln\left(1-\frac{\alpha(H)}{m}\right)}.
\end{aligned}$$ □

The following lemma is a standard lemma in graph theory (see, *e.g.*, [7, p. 147]).

Lemma 6.3.3. *Given any hypergraph $H = (S, \mathcal{E})$, $\chi(H)\alpha(H) \geq |V(H)|$.*

Proof. Colour the vertices of H with $\chi(H)$ colours so that H has no monochromatic edge. Let $C_1, C_2, ..., C_{\chi(H)}$ be the colour classes formed by this colouring. Then

$$|V(H)| = \sum_{i=1}^{\chi(H)} |C_i| \leq \sum_{i=1}^{\chi(H)} \alpha(H) = \chi(H)\alpha(H).$$ □

Corollary 6.3.4. *For $H = (S, \mathcal{E})$, a symmetric hypergraph with $m = |S|$, and $\mathcal{E} \neq \emptyset$,*

$$\frac{m}{\alpha(H)} \leq \chi(H) < 1 + \frac{m}{\alpha(H)}\ln m.$$

[The upper bound in Corollary 6.3.4 is slightly different than that given in the source (equation (26), p. 100).]

Proof. The lower bound is exactly Lemma 6.3.3. The upper bound is an estimation of the upper bound given in the Symmetric Hypergraph theorem. Recall that a consequence of the fact that for all $x \in (0,1)$, $e^{-x} > 1-x$ is that for $x \in (0,1)$,

$$x < -\ln(1-x). \tag{6.4}$$

Therefore,

$$\begin{aligned}
\chi(H) &\leq 1 + \frac{\ln m}{-\ln(1-\frac{\alpha(H)}{m})} && \text{(by Theorem 6.3.2)},\\
&< 1 + \frac{\ln m}{\left(\frac{\alpha(H)}{m}\right)} && \text{(by equation (6.4) with } x = \tfrac{\alpha(H)}{m}),\\
&= 1 + \frac{m}{\alpha(H)} \ln m.
\end{aligned}$$

□

6.3.3 Application to graph Ramsey numbers

What follows is a connection between the graph Ramsey numbers and the extremal numbers.

Let $\mathcal{H} = \{H_m\}_{m=1}^{\infty}$ be a sequence of symmetric hypergraphs $H_m = (S_m, \mathcal{E}_m)$ with the properties that for all $m \in \mathbb{Z}^+$, $\mathcal{E}_m \neq \emptyset$, and $\frac{|S_m|}{\alpha(H_m)}$ is a strictly increasing function in m. Let $f(m) = \frac{|S_m|}{\alpha(H_m)}$, and let $g(m) = 1 + f(m)\ln|S_m|$. Then by Corollary 6.3.4, for all $m \in \mathbb{Z}^+$,

$$f(m) \leq \chi(H_m) < g(m). \tag{6.5}$$

Lemma 6.3.5. *Let $\mathcal{H}, f, g$ be as above, and let $t \in \mathbb{Z}^+$. Then there exists a least*

$M = \mathcal{R}_{\mathcal{H}}(t)$ satisfying

$$g^{-1}(t) < \mathcal{R}_{\mathcal{H}}(t) \leq 1 + f^{-1}(t)$$

such that for all $m \geq M$, $\chi(H_m) > t$.

Proof. (Upper bound) Let $x \geq f^{-1}(t) + 1$. Then

$$\begin{aligned}
\chi(H_x) &\geq f(x) && \text{(by the left inequality in (6.5))} \\
&\geq f(f^{-1}(t) + 1) && \text{(since } f \text{ is increasing)} \\
&> f(f^{-1}(t)) && \text{(since } f \text{ is strictly increasing)} \\
&= t.
\end{aligned}$$

Therefore $\mathcal{R}_{\mathcal{H}}(t) \leq f^{-1}(t) + 1$.

(Lower bound) Let $M = \mathcal{R}_{\mathcal{H}}(t)$. Then

$$\begin{aligned}
g(M) &> \chi(H_M) && \text{(by the upper bound in (6.5))} \\
&> t && \text{(by the definition of } \mathcal{R}_{\mathcal{H}}(t)).
\end{aligned}$$

Since f is increasing, g is also increasing, and therefore $g(M) > t$ iff $\mathcal{R}_{\mathcal{H}}(t) = M > g^{-1}(t)$. □

Let G be a graph. For each $m \in \mathbb{Z}^+$, let $S_m = E(K_m)$, and

$$\mathcal{E}_m = \left\{ E(G') : G' \in \binom{K_m}{G} \right\}.$$

Let $\mathcal{H} = \{H_m\}_{m=1}^{\infty}$ be the sequence of hypergraphs $H_m = (S_m, \mathcal{E}_m)$. Then $\alpha(H_m) = ex(m, G)$.

Example. Let $G = K_3$, and $m = 4$. Then the vertices in H_4 are edges of K_4, with edges copies of K_3 inside the K_4. If the vertices of the K_4 were $\{a, b, c, d\}$, then

$$S_4 = \{\{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}\},$$

$$\mathcal{E}_4 = \{\{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\}.$$

Theorem 6.3.6. *For any $t \in \mathbb{Z}^+$, $\mathcal{R}_{\mathcal{H}}(t) = R(G; K_2; t)$.*

Proof. The proof is just a matter of unravelling the definitions.

$$\begin{aligned}
\mathcal{R}_{\mathcal{H}}(t) &= \text{the least } M \text{ s.t. } \forall m \geq M,\ \chi(H_m) > t \\
&= \text{the least } M \text{ s.t. } \forall m \geq M,\ (\text{and a hyperedge } e_m),\ H_m \to (e_m)_t^{K_1} \\
&= \text{the least } M \text{ s.t. } \forall m \geq M,\ K_m \to (G)_t^{K_2}. \qquad (\text{by the definition of } \mathcal{H}) \\
&= R(G; K_2; t).
\end{aligned}$$

□

Corollary 6.3.7. *Fix a graph G. If*

$$f(m) = \frac{\binom{m}{2}}{ex(m, G)} \qquad \textit{and} \qquad g(m) = 1 + f(m) \ln \binom{m}{2}$$

are strictly increasing functions in m, then $g^{-1}(t) < R(G; K_2; t) \leq f^{-1}(t) + 1$.

For example, when $G = C_4$, it is known (see, *e.g.*, [13] and [47]) that for some constant c_1,

$$ex(m, C_4) \sim c_1 n^{3/2}.$$

Proposition 6.3.8 (Graham, Rothschild and Spencer, 1990 [65])**.** *There exist constants c_4, c_5 such that for all $t \in \mathbb{Z}^+$, $t \geq 2$,*

$$\frac{c_5 t^2}{(\ln t)^2}(1 + o(1)) < R(C_4; K_2; t) \leq c_4 t^2.$$

Proof. (as presented in [65]) Since $ex(m, C_4) \sim c_1 n^{3/2}$, for some constants c_2 and c_3,

$$f(m) = \frac{\binom{m}{2}}{ex(m, G)} = c_2 m^{1/2}(1 + o(1))$$

and

$$g(m) = 1 + f(m) \ln \binom{m}{2} = c_3 m^{1/2} \ln m(1 + o(1))$$

Both f and g are strictly increasing functions, and so the bounds in Corollary 6.3.7 hold. The inverse of f is, for some constant c_4, $f^{-1}(t) = c_4 t^2$, which together with Corollary 6.3.7, proves the lower bound. Finding the inverse of g is not as easy. Let $W(y)$ be the inverse of the function $y = xe^x$, *i.e.*,

$$y = xe^x \qquad \Longleftrightarrow \qquad x = W(y).$$

The function $W(y)$ is known as the Lambert W function (see, *e.g.*, [31]).

Claim 1. $g^{-1}(y) = e^{2W(\frac{y}{2c_3})}$.

Proof of claim 1. The proof of Claim 1 is by the following algebra:

$$y \quad = \quad c_3 \sqrt{m} \ln m,$$

$$\begin{aligned}
&= c_3 e^{\ln\sqrt{m}} \ln m, \\
\frac{1}{2}y &= c_3 e^{\ln\sqrt{m}} \left(\frac{1}{2}\ln m\right), \\
&= c_3 e^{\ln\sqrt{m}} (\ln\sqrt{m}), \\
\frac{1}{2c_3}y &= ue^u \qquad\qquad (\text{letting } u = \ln\sqrt{m}), \\
W\left(\frac{y}{2c_3}\right) &= u, \\
&= ln\sqrt{m}, \\
e^{W\left(\frac{y}{2c_3}\right)} &= \sqrt{m}, \\
e^{2W\left(\frac{y}{2c_3}\right)} &= m,
\end{aligned}$$

which proves the claim.

To prove the upper bound in Proposition 6.3.8, by Corollary 6.3.7, it now suffices to prove the following claim.

Claim 2. For all $y \in \mathbb{R}^+$, $W(y) = (1 + o(1))(\ln y - \ln\ln y)$, and therefore for some constant c_5,

$$g^{-1}(t) = (1 + o(1))\frac{c_5 t^2}{(\ln t)^2}.$$

Proof of claim 2. Let ϕ be the real-valued function $\phi : x \mapsto xe^x$. Then

$$\begin{aligned}
\phi^{-1} = W(y) &= (1 + o(1))(\ln y - \ln\ln y), \\
\text{iff} \quad \phi\left((1 + o(1))(\ln y - \ln\ln y)\right) &= (1 + o(1))y, \\
\text{iff} \quad (1 + o(1))(\ln y - \ln\ln y)e^{(1+o(1))(\ln y - \ln\ln y)} &= (1 + o(1))y,
\end{aligned}$$

$$\begin{aligned}
\text{iff} \quad (1+o(1))(\ln y - \ln\ln y)y\frac{1}{\ln y} &= (1+o(1))y, \\
\text{iff} \quad (1+o(1))\left(1 - \frac{\ln\ln y}{\ln y}\right) y &= (1+o(1))y,
\end{aligned}$$

which holds. Therefore

$$W(y) = (1+o(1))(\ln y - \ln\ln y). \tag{6.6}$$

Then the closed form for $g^{-1}(y)$ is found as follows:

$$\begin{aligned}
g^{-1}(y) &= (1+o(1))e^{2W\left(\frac{y}{2c_3}\right)} && \text{(by Claim 1)}, \\
&= (1+o(1))e^{2\left(\ln\frac{y}{2c_3} - \ln\ln\frac{y}{2c_3}\right)} && \text{(by equation (6.6))}, \\
&= (1+o(1))\left(\frac{y}{2c_3}\right)^2\left(\ln\left(\frac{y}{2c_3}\right)\right)^{-2}, \\
&= (1+o(1))\left(\frac{y}{2c_3\ln\left(\frac{y}{2c_3}\right)}\right)^2, \\
&= (1+o(1))c_5\left(\frac{y}{\ln y}\right)^2,
\end{aligned}$$

which proves Claim 2, and the proposition. □

Chapter 7

Linear Ramsey theory

7.1 Introduction

The diagonal graph Ramsey numbers for complete graphs grow exponentially in terms of the number of vertices, but not all graph Ramsey numbers grow nearly this fast. For instance, a consequence of Theorem 4.4.2 is that the graph Ramsey numbers for trees are, at worst, quadratic in the number of vertices. A family $\mathcal{F}$ of graphs is *linear Ramsey* iff there exists a constant $c = c(\mathcal{F})$ such that for all $G \in \mathcal{F}$, $R(G, G) \leq c|V(G)|$.

In one of the earliest papers on the topic of Linear Ramsey theory, Burr and Erdős presented the following two examples of linear Ramsey families.

Theorem 7.1.1 (Burr and Erdős, 1973 [16]). *For any graph G, the family*

$$\mathcal{F} = \{G, 2G, 3G, \ldots\}$$

is linear Ramsey with $c(\mathcal{F}) = R(G,G) + 2$.

Proof. Fix a graph G, and let $\mathcal{F} = \{G, 2G, 3G, \ldots\}$. Set $c = c(\mathcal{F}) = R(G,G) + 2$. Then, for any positive integer k,

$$\begin{aligned} R(kG, kG) &\leq R(G,G) + (k-1)|V(G)| + (k-1)|V(G)| \qquad \text{(by Theorem 4.4.3)} \\ &= \left(\frac{R(G,G)}{k|V(G)|} + 2 - \frac{2}{k}\right) k|V(G)| \\ &< (R(G,G)+2)|V(kG)| \\ &= c|V(kG)|. \end{aligned}$$

□

Theorem 7.1.2 (Burr and Erdős, 1973 [16]). *The family $\mathcal{F} = \{4K_1, 4^2K_2, 4^3K_3, \ldots\}$ is linear Ramsey with $c(\mathcal{F}) = 3$.*

Proof. By Corollary 3.2.2 (the upper bound on the traditional Ramsey numbers), for any $i \in \mathbb{Z}^+$, $R(K_i, K_i) \leq 4^i$. Therefore for any $i \in \mathbb{Z}^+$,

$$\begin{aligned} R(4^iK_i, 4^iK_i) &\leq R(K_i, K_i) + (4^i - 1)i + (4^i - 1)i \qquad \text{(by Theorem 4.4.3)} \\ &\leq 4^i + 2i(4^i - 1) \\ &= \left(\frac{1}{i} + 2 - \frac{2}{4^i}\right) i4^i \\ &< 3|V(4^iK_i)|. \end{aligned}$$

□

7.2 Bounded maximum degree implies linear

The main result of this section is a result of Chvátal, Rödl, Szemerédi and Trotter, who proved in 1983 [28] that for any $d \in \mathbb{Z}^+$, the family of all graphs with max degree less than or equal to d is linear Ramsey. The proof presented here is found in Diestel's book [33], which is based on the original proof by Chvátal *et al.* For this proof, three theorems in graph theory are used: Szemerédi's Regularity lemma, Turán's theorem (given already as Theorem 6.2.1), and the familiar finite version of Ramsey's theorem (Theorem 1.2.3).

Throughout this section, G and H are graphs, and $\varepsilon > 0$.

7.2.1 Szemerédi's Regularity lemma

Let $X, Y \subseteq V(G)$, $X \cap Y = \emptyset$. Let $E_G(X,Y)$ denote the set of edges of G between X and Y, *i.e.*,

$$E_G(X,Y) = E(G) \cap \{\{x,y\} : x \in X, y \in Y\}.$$

The *edge density between X and Y* is

$$\rho_G(X,Y) = \frac{|E_G(X,Y)|}{|X||Y|}$$

(when it is clear which graph the density is in, the G may be omitted). For $\varepsilon > 0$, the pair (X,Y) is said to be *ε-regular* iff for all $A \subseteq X$, $B \subseteq Y$ satisfying $|A| \geq \varepsilon|X|$,

$|B| \geq \varepsilon|Y|$,

$$|\rho(A, B) - \rho(X, Y)| \leq \varepsilon.$$

A partition $V(G) = V_0 \cup V_1 \cup \cdots \cup V_k$ is *ε-regular* if $0 \leq |V_0| \leq \varepsilon|V(G)|$, $|V_1| = \cdots = |V_k|$, and at most $\varepsilon\binom{k}{2}$ pairs of the form (V_i, V_j), $1 \leq i < j \leq k$, are not ε-regular. A graph G is *ε-regular* if it admits an ε-regular partition.

Lemma 7.2.1. *Let $\varepsilon_0, \varepsilon_1$ be such that $0 < \varepsilon_0 \leq \varepsilon_1 \leq 1$. If P is an ε_0-regular partition $V(G) = V_0 \cup V_1 \cup \cdots \cup V_k$, then P is also an ε_1-regular partition of $V(G)$. Therefore, every ε_0-regular graph is also ε_1-regular.*

Proof. Let P be an ε_0-regular partition $V(G) = V_0 \cup V_1 \cup \cdots \cup V_k$. Let $i, j \in \mathbb{Z}^+$, $i < j$, be such that (V_i, V_j) is ε_0-regular. Let $A \subseteq V_i$ and $B \subseteq V_j$ satisfy $|A| \geq \varepsilon_1|V_i|$, $|B| \geq \varepsilon_1|V_j|$. Then $|A| \geq \varepsilon_1|V_i| \geq \varepsilon_0|V_i|$, and $|B| \geq \varepsilon_1|V_j| \geq \varepsilon_0|V_j|$, and since (V_i, V_j) is ε_0-regular,

$$|\rho(A, B) - \rho(V_i, V_j)| \leq \varepsilon_0 \leq \varepsilon_1,$$

which shows (V_i, V_j) is ε_1-regular. Therefore every ε_0-regular pair (V_i, V_j) is also ε_1-regular, and each pair (V_i, V_j) that is not ε_1-regular is also not ε_0-regular. Thus the number of pairs (V_i, V_j) that are not ε_1-regular is at most the number that are at not ε_0-regular, which is at most $\varepsilon_0\binom{k}{2} \leq \varepsilon_1\binom{k}{2}$. □

Note that ε is used a number of times in the above definitions, and so the definitions could have been made using "different ε's". One could, for example, define the pair

(X, Y) to be $(\varepsilon_1, \varepsilon_2)$-regular iff for all $A \subseteq X$, $B \subseteq Y$ satisfying $|A| \geq \varepsilon_1|X|$ $B \geq \varepsilon_1|Y|$,

$$|\rho(A, B) - \rho(X, Y)| \leq \varepsilon_2,$$

and one could define a partition $V(G) = V_0 \cup V_1 \cup \cdots \cup V_k$ to be $(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)$-regular if $0 \leq |V_0| \leq \varepsilon_3|V(G)|$, $|V_1| = \cdots = |V_k|$, and at most ε_4-regular pairs of the form (V_i, V_j), $1 \leq i < j \leq k$ are not $(\varepsilon_1, \varepsilon_2)$-regular. However, for the purposes of this work, all the ε's are taken to be the same.

Lemma 7.2.2. *For any $A, B \subseteq V(G)$ with $A \cap B = \emptyset$, $\rho_G(A, B) = 1 - \rho_{\overline{G}}(A, B)$.*

Proof. By the definition of $\rho_G(A, B)$,

$$\begin{aligned} \rho_G(A, B) &= \frac{|E_G(A, B)|}{|A||B|} \\ &= \frac{|A||B| - |E_{\overline{G}}(A, B)|}{|A||B|} \\ &= 1 - \frac{|E_{\overline{G}}(A, B)|}{|A||B|} \\ &= 1 - \rho_{\overline{G}}(A, B). \end{aligned}$$

□

Lemma 7.2.3. *For any disjoint sets $V_1, V_2 \subseteq V(G)$ such that (V_1, V_2) is ε-regular in G, the pair (V_1, V_2) is also ε-regular in $\overline{G}$. Therefore, if G is ε-regular, admitting an ε-regular partition P, then $\overline{G}$ is also ε-regular, witnessed by the same partition.*

Proof. Let $V_1, V_2 \subseteq V(G)$ be disjoint sets such that (V_1, V_2) is an ε-regular pair in G. Let $A \subseteq V_i$, $B \subseteq V_j$ with $|A| \geq \varepsilon|V_i|$, $|B| \geq \varepsilon|V_j|$. Then

$$|\rho_{\overline{G}}(A, B) - \rho_{\overline{G}}(V_i, V_j)| = |(1 - \rho_G(A, B)) - (1 - \rho_G(V_i, V_j))| \quad \text{(by Lemma 7.2.2)}$$

$$\begin{aligned} &= |\rho_G(V_i, V_j) - \rho_G(A, B)| \\ &\leq \varepsilon \qquad\qquad \text{(since } (V_i, V_j) \text{ is } \varepsilon\text{-regular in } G). \end{aligned}$$ □

Lemma 7.2.4. *Let G be a graph containing an ε-regular pair (A, B). Let $d = \rho(A, B)$, and let $Y \subseteq B$, $|Y| \geq \varepsilon|B|$. Then more than $(1 - \varepsilon)|A|$ vertices in A have at least $(d - \varepsilon)|Y|$ neighbours in Y.*

Proof. Let X be the set of vertices in A with fewer than $(d - \varepsilon)|Y|$ neighbours in Y. Then

$$\rho(X, Y) - d < \frac{|X|(d - \varepsilon)|Y|}{|X||Y|} - d = -\varepsilon.$$

Therefore $\rho(X, Y) - \rho(A, B) < -\varepsilon$. Since (A, B) is ε-regular, and $|Y| \geq \varepsilon|B|$, if it were the case that $|X| \geq \varepsilon|A|$, then

$$|\rho(X, Y) - \rho(A, B)| \leq \varepsilon,$$

which is not true. Therefore $|X| < \varepsilon|A|$. □

Szemerédi published his “regularity lemma” in 1978 [138] after using a bipartite version of the regularity lemma in order to prove a conjecture of Erdős and Turán [49] relating to arithmetic progressions in the integers. For $k, n \in \mathbb{Z}^+$, let $r_k(n)$ denote the greatest integer ℓ such that there exists $0 < a_1 < a_2 < \ldots < a_\ell \leq n$ such that $\{a_1, \ldots, a_\ell\}$ does not contain an arithmetic progression of k terms. The conjecture of Erdős and Turán [49], which Szemerédi [137] proved, was that for all $k \in \mathbb{Z}^+$, $r_k(n)$ is $o(n)$ (that is, $\lim_{n\to\infty} \frac{r_k(n)}{n} = 0$).

The regularity lemma has been restated in many different forms over the years, and the following form is found in [33].

Lemma 7.2.5 (Szemerédi's Regularity lemma)**.** *For every $\varepsilon > 0$, and $m \in \mathbb{Z}^+$, there exists $M = M(\varepsilon, m) \in \mathbb{Z}^+$, such that for every graph G with at least m vertices, there exists $k \in [m, M]$ such that G admits an ε-regular partition $V(G) = V_0 \cup V_1 \cup \cdots \cup V_k$.*

The proof of the regularity lemma is omitted, and may be found in, *e.g.*, [7] and [33]. The interested reader can see [84] and [85] for detailed surveys on applications of Szemerédi's Regularity lemma in graph theory.

7.2.2 Regularity graphs

For a graph G with ε-regular partition $P : V(G) = V_0 \cup V_1 \cup \cdots \cup V_k$, and $d \in \mathbb{R}$, $0 \le d \le 1$, the *regularity graph for G (with partition P and parameter d)*, denoted $\mathrm{REG}^P(G; d)$, is the graph with vertex set $\{V_1, \ldots, V_k\}$ and edge set

$$\{\{V_i, V_j\} : \ (V_i, V_j) \text{ is } \varepsilon\text{-regular, and } \rho(V_i, V_j) \ge d\}.$$

When it is clear which partition is used, $\mathrm{REG}^P(G; d)$ may be written as simply $\mathrm{REG}(G; d)$. For any $s \in \mathbb{Z}^+$, let $\mathrm{REG}_s^P(G; d)$ denote the graph formed from

$\mathrm{REG}^P(G;d)$ by replacing each vertex in $REG^P(G;d)$ with s independent vertices, and each edge by a complete bipartite graph $K_{s,s}$.

The following Lemma proves a sufficient condition for a graph H with bounded max degree to be found in an ε-regular graph G, which is used to prove the main result that the set of graphs with bounded max degree is linear Ramsey.

Lemma 7.2.6. *For all $D \geq 1$, there exists $\varepsilon_0 = \varepsilon_0(D) > 0$, such that for any graph G admitting an ε_0-regular partition $V(G) = V_0 \cup V_1 \cup \cdots \cup V_k$, $\ell = |V_1| = \cdots = |V_k|$, any graph H with $\Delta(H) \leq D$, and for any positive integer $s \leq \frac{\ell}{2^{D+1}}$, if $H \subseteq REG_s(G; \frac{1}{2})$, then $H \subseteq G$.*

Proof. Let $D \geq 1$. Choose $\varepsilon_0 > 0$ small enough so that $\varepsilon_0 < \frac{1}{2}$, and

$$\left(\frac{1}{2} - \varepsilon_0\right)^D - D\varepsilon_0 \;\geq\; \left(\frac{1}{2}\right)^{D+1}$$

(such a choice is possible since as $\varepsilon_0 \to 0$, $\left(\frac{1}{2} - \varepsilon_0\right)^D - D\varepsilon_0 \to \left(\frac{1}{2}\right)^D$).

Let G be an ε_0-regular graph admitting the ε_0-regular partition $V(G) = V_0 \cup V_1 \cup \cdots \cup V_k$. Let $\ell = |V_1| = \cdots = |V_k|$, and let $s \leq \frac{\ell}{2^{D+1}}$. Let H be any graph with $\Delta(H) \leq D$, and assume that $H \subseteq \mathrm{REG}_s(G; \frac{1}{2})$ (see Figure 7.1).

Note that $H \subseteq \mathrm{REG}_s(G; \frac{1}{2})$ implies that $s > 0$. Enumerate $V(H) = \{u_1, \ldots, u_h\}$. The graph H is embedded in G with the embedding defined inductively as follows. Let $\sigma : [1,h] \to [1,k]$ be such that for all $i \in [1,h]$, u_i is in the set of s vertices in

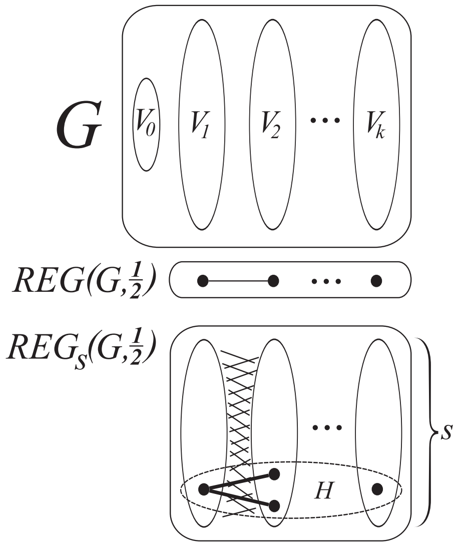


Figure 7.1: Definition of $REG(G;\frac{1}{2})$ and $REG_s(G;\frac{1}{2})$

$REG_s(G;\frac{1}{2})$ that replaced $V_{\sigma(i)}$ in $REG(G;\frac{1}{2})$.

For each $i \in [1,h]$, let

$$Y_i^0 = V_{\sigma(i)}.$$

For any $n \in [0,h]$, let $H_n = H[\{u_1,\ldots,u_n\}]$, and let $J(n)$ be the statement that there exists an embedding f of H_n into G, and that for all $i \in [1,h]$, the sets $Y_i^n \subseteq V_{\sigma(i)}$ have been defined and satisfy

$$|Y_i^n| \geq \ell\left(\frac{1}{2}-\varepsilon_0\right)^{\deg_{H_n}(u_i)}$$

(note that H_0 is accepted to be the null graph, the graph containing no vertices and no edges). To prove Lemma 7.2.6, it suffices to show that $J(h)$ holds.

Base Case: Since the null graph is (trivially) a subgraph of every graph, and since

$|Y_i^0| = |V_{\sigma(i)}| = \ell$, the statement $J(0)$ holds.

Inductive Step: Let $m \in [0, h-1]$, and assume that $J(m)$ holds. Let f be an embedding of H_m into G.

Then for all $i \in [1, h]$,

$$\begin{aligned}
|Y_i^m| &\geq \ell\left(\frac{1}{2} - \varepsilon_0\right)^{\deg_{H_m}(u_i)} && \text{(since } J(m) \text{ holds)} \\
&\geq \ell\left(\frac{1}{2} - \varepsilon_0\right)^{D} && \text{(since } \deg_{H_m}(u_i) \leq \Delta(H) \leq D) \\
&= \ell\left[\left(\frac{1}{2} - \varepsilon_0\right)^{D} - D\varepsilon_0\right] + \ell D\varepsilon_0 \\
&\geq \ell\frac{1}{2^{D+1}} + \ell D\varepsilon_0 && \text{(by the choice of } \varepsilon_0) \\
&\geq s + \ell D\varepsilon_0 && \text{(by definition of } s). \qquad (7.1)
\end{aligned}$$

Let $I = \{i > m : \{u_{m+1}, u_i\} \in E(H)\}$. For all $i > m$, $i \notin I$, let $Y_i^{m+1} = Y_i^m$. Fix some $i \in I$. Then let

$$\begin{aligned}
A &= V_{\sigma(m+1)}, \\
B &= V_{\sigma(i)}, \\
Y &= Y_i^m.
\end{aligned}$$

By equation (7.1), $|Y| = |Y_i^m| \geq s + D\varepsilon_0\ell > \varepsilon_0\ell$, and so Lemma 7.2.4 can be applied to yield a set of more than $(1-\varepsilon_0)|A|$ vertices in A that each have at least $(\frac{1}{2} - \varepsilon_0)|Y|$ neighbours in Y.

Since $|I| \leq D$, applying Lemma 7.2.4 repeatedly (for each $i \in I$) implies the existence of a set of more than $(1 - D\varepsilon_0)|A|$ vertices in A (and thus more than $|Y^m_{m+1}| - D\varepsilon_0|A|$ vertices in $Y^m_{m+1} \subseteq A$) that each have at least $(\frac{1}{2} - \varepsilon_0)|Y^m_i|$ neighbours in each Y^m_i. Then since

$$|Y^m_{m+1}| - D\varepsilon_0|A| = |Y^m_{m+1}| - \ell D\varepsilon_0 \geq s > 0 \qquad \text{(by equation (7.1))},$$

there is at least one vertex v_{m+1} left in Y^m_{m+1} with the desired properties. Extend f to f^* by defining $f^*(u_{m+1}) = v_{m+1}$.

Then for all $i \in I$, let $Y^{m+1}_i = Y^m_i \cap N_G(v_{m+1})$. Then

$$\begin{aligned}
|Y^{m+1}_i| &\geq \left(\frac{1}{2} - \varepsilon_0\right)|Y^m_i| \\
&\geq \ell\left(\frac{1}{2} - \varepsilon_0\right)^{\deg_{H_m}(u_i)+1} \qquad \text{(since } J(m) \text{ holds)} \\
&= \ell\left(\frac{1}{2} - \varepsilon_0\right)^{\deg_{H_{m+1}}(u_i)}.
\end{aligned}$$

Therefore for all $i \in [1, h]$,

$$|Y^{m+1}_i| \geq \ell\left(\frac{1}{2} - \varepsilon_0\right)^{\deg_{H_{m+1}}(u_i)},$$

and so $J(m + 1)$ holds, finishing the inductive step.

Thus by mathematical induction, for all $n \in [0, h]$, $J(n)$ holds. Specifically, $J(h)$ holds, which proves the lemma. $\square$

7.2.3 Chvátal *et al.*'s proof

The main result of this chapter can now be stated and proved.

Theorem 7.2.7 (Chvátal, Rödl, Szemerédi and Trotter, 1983 [28]). *For all $d \geq 1$, there exists $c = c(d)$ such that for all graphs H with $\Delta(H) \leq d$, $R(H, H) \leq c|V(H)|$.*

Proof. For any $D \geq 1$, $\varepsilon > 0$, let $S(D, \varepsilon)$ be the statement that for any graph G admitting an ε-regular partition $V(G) = V_0 \cup V_1 \cup \cdots \cup V_k$, $\ell = |V_1| = \cdots = |V_k|$, for any graph H with $\Delta(H) \leq D$, and for any positive integer $s \leq \frac{\ell}{2^{D+1}}$, if $H \subseteq REG_s(G; \frac{1}{2})$, then $H \subseteq G$.

Fix $D \geq 1$. By Lemma 7.2.6, there exists $\varepsilon_0 > 0$, $\varepsilon_0 = \varepsilon_0(D)$, such that $S(D, \varepsilon_0)$ holds. Let $m = R(D+1, D+1)$, the Ramsey number, and let $\varepsilon > 0$ be such that

$$\varepsilon < \min \left\{ \varepsilon_0, \frac{1}{m(m-1)} \right\}.$$

By Szemerédi's Regularity lemma (Lemma 7.2.5), there exists $M = M(\varepsilon, m) \in \mathbb{Z}^+$, such that for every graph G on at least m vertices, there exists a $k \in [m, M]$ such that G admits an ε-regular partition $V(G) = V_0 \cup V_1 \cup \cdots \cup V_k$. Let $c = \left\lceil \frac{2^{D+1}M}{1-\varepsilon} \right\rceil$.

Let H be any graph with $\Delta(H) \leq D$. Let $s = |V(H)|$, and let $n = cs$. Two-colour the edges of K_n with green and blue, and let G be the green graph (and $\overline{G}$ the blue graph).

Since

$$|V(G)| = n = cs \geq M \geq m,$$

the regularity lemma implies that there exists $k \in [m, M]$ and an ε-regular partition $V(G) = V_0 \cup V_1 \cup \cdots \cup V_k$. Let $\ell = |V_1| = \cdots = |V_k|$. The number of edges in $\mathrm{REG}(G; 0)$ is at least

$$\begin{aligned}
(1-\varepsilon)\binom{k}{2} &= \frac{1}{2}k^2\left(\frac{k-1}{k}\right)(1-\varepsilon) \\
&= \frac{1}{2}k^2\left[1 - \frac{1}{k} - \left(\frac{k-1}{k}\right)\varepsilon\right] \\
&> \frac{1}{2}k^2\left(1 - \frac{1}{k} - \varepsilon\right) && \left(\text{since } \frac{k-1}{k} < 1\right) \\
&> \frac{1}{2}k^2\left(1 - \frac{1}{k} - \frac{1}{m(m-1)}\right) && \left(\text{since } \varepsilon < \frac{1}{m(m-1)}\right) \\
&= \frac{1}{2}k^2\left(1 - \frac{1}{k} - \frac{1}{m-1} + \frac{1}{m}\right) \\
&\geq \frac{1}{2}k^2\left(1 - \frac{1}{k} - \frac{1}{m-1} + \frac{1}{k}\right) && (\text{since } m \leq k) \\
&= \frac{1}{2}k^2\left(1 - \frac{1}{m-1}\right) \\
&= \frac{1}{2}k^2\left(\frac{m-2}{m-1}\right) \\
&\geq t(k, m-1) && (\text{by Theorem 6.2.3}).
\end{aligned}$$

Therefore by Turán's theorem (Theorem 6.2.1), there is a copy of K_m, call it K'_m, inside $\mathrm{REG}(G; 0)$.

Colour the edges of K'_m red and white as follows: colour $\{V_i, V_j\}$ red if $\rho(V_i, V_j) \geq \frac{1}{2}$, and white if $\rho(V_i, V_j) < \frac{1}{2}$. Since $m = R(D+1, D+1)$, there is a monochromatic copy of K_{D+1} in K'_m, call it K'_{D+1}.

Claim 1. $s \leq \frac{\ell}{2^{D+1}}$.

Proof of Claim 1.

$$\begin{aligned}
\ell &= |V_1| = \cdots = |V_k| \\
&= \frac{n - |V_0|}{k} \\
&\geq \frac{n - \varepsilon n}{k} && \text{(since } |V_0| \leq \varepsilon n\text{)} \\
&\geq n\left(\frac{1-\varepsilon}{M}\right) && \text{(since } k \leq M\text{)} \\
&= cs\left(\frac{1-\varepsilon}{M}\right) && \text{(by the definition of } n\text{)} \\
&= \left(\frac{2^{D+1}M}{1-\varepsilon}\right)\left(\frac{1-\varepsilon}{M}\right)s && \text{(by the definition of } c\text{)} \\
&= 2^{D+1}s,
\end{aligned}$$

and Claim 1 is proved.

If K'_{D+1} is red, then $K'_{D+1} \subseteq \text{REG}(G; \frac{1}{2})$. Apply $S(D, \varepsilon_0)$ with G, H, and s to get that if $H \subseteq \text{REG}_s(G; \frac{1}{2})$, then $H \subseteq G$ (the requirements on s in $S(D, \varepsilon_0)$ hold by Claim 1). But,

$$\begin{aligned}
H &\subseteq K_{\underbrace{s, \ldots, s}_{\chi(H)}} && \text{(since } s = |V(H)|\text{)} \\
&\subseteq K_{\underbrace{s, \ldots, s}_{D+1}} && \text{(since } \chi(H) \leq \Delta(H) + 1 \leq D+1\text{)}
\end{aligned}$$

$$\subseteq \quad \mathrm{REG}_s\left(G; \frac{1}{2}\right) \qquad\qquad \left(\text{since } K'_{D+1} \subseteq \mathrm{REG}\left(G; \frac{1}{2}\right)\right).$$

Therefore $H \subseteq G$.

If K'_{D+1} is white, then since the ε-regular partition of G is also an ε-regular partition of $\overline{G}$ (by Lemma 7.2.3), applying $S(D, \varepsilon_0)$ with $\overline{G}$, H, and s, by the same logic as above since $K'_{D+1} \subseteq \mathrm{REG}\left(\overline{G}; \frac{1}{2}\right)$, it follows that $H \subseteq \overline{G}$. □

In 2000, Graham, Rödl, and Ruciński [62] published a proof to Theorem 7.2.7 avoiding the use of the regularity lemma, and improving the constant $c = c(d)$. The same authors published a year later [63] that, for bipartite graphs of max degree, the constant could be further improved.

7.3 A Burr-Erdős conjecture

In their 1973 paper, Burr and Erdős presented a conjecture in the field of Linear Ramsey theory that is still open today. Recall (see Appendix) that a *forest* is an a-cyclic graph. Given a graph G, the *arboricity of* G, denoted $\Upsilon(G)$, is the minimum number r so that there exists a partition $E(G) = E_1 \cup \cdots \cup E_r$ such that for all $i \in [1, r]$, the graph $G_i = (V(G), E_i)$ is a forest. For example, $\Upsilon(K_4) = 2$ since K_4 can be separated into two P_3's, and $\Upsilon(K_5) = 3$ since K_5 can be separated into two

$P_3 \dot{\cup} K_1$, and one $K_{1,4}$.

Conjecture 7.3.1 (Burr, Erdős 1973 [16]). *For any $d \in \mathbb{Z}^+$, the set of graphs with arboricity at most d is linear Ramsey.*

Define the *edge density* of a graph G by

$$\rho(G) = \max_{F \subseteq G} \frac{|E(F)|}{|V(F)|}.$$

Recall that for any graph G, $\delta(G)$ denotes the minimum degree over all vertices of G. For any $k \in \mathbb{Z}^+$, a graph is *k-degenerate* iff every subgraph of G contains a vertex of degree less than or equal to k. Alternatively, G is k-degenerate iff $\max\{\delta(F) : F \subseteq G\} \leq k$. For example, for any fixed $n \in \mathbb{Z}^+$, K_n is n-degenerate, and $(n-1)$-degenerate, but not $(n-2)$-degenerate. The *degeneracy number* of a graph G, denoted $\sigma(G)$, is the least k such that G is k-degenerate.

Theorem 7.3.2 (Burr and Erdős, 1973 [16]). *For any graph G,*

$$\frac{1}{2}\sigma(G) \leq \rho(G) < \Upsilon(G) + 1 \leq 2\sigma(G) + 1.$$

Proof. Let G be a graph. To see the first inequality,

$$\begin{aligned}
\frac{1}{2}\sigma(G) &= \frac{1}{2}\max_{F \subseteq G} \delta(F) \\
&\leq \frac{1}{2}\max_{F \subseteq G} \frac{1}{|V(F)|}\sum_{v \in V(F)} \deg(v) \qquad \left(\text{since } \delta(F) \leq \frac{1}{|V(F)|}\sum_{v \in V(F)} \deg(v)\right) \\
&= \frac{1}{2}\max_{F \subseteq G} \frac{2|E(F)|}{|V(F)|}
\end{aligned}$$

$$= \rho(G).$$

For the second inequality in the statement of Theorem 7.3.2, as pointed out by Harary [72, p.90], Nash-Williams proved that for any graph G,

$$\Upsilon(G) = \max_{F \subseteq G} \left\lfloor \frac{|E(F)|}{|V(F)|-1} \right\rfloor. \tag{7.2}$$

Therefore,

$$\rho(G) = \max_{F \subseteq G} \frac{|E(F)|}{|V(F)|} < \max_{F \subseteq G} \frac{|E(F)|}{|V(F)|-1} \le \max_{F \subseteq G} \left\lfloor \frac{|E(F)|}{|V(F)|-1} \right\rfloor + 1 = \Upsilon(G) + 1.$$

To prove the third inequality in the statement of Theorem 7.3.2, first note that if G is empty, the inequality is trivially true, so assume for some non-empty graph G that the inequality failed; that is, assume $\Upsilon(G) > 2\sigma(G)$. Then by equation (7.2), there exists $F \subseteq G$ such that $\frac{|E(F)|}{|V(F)|-1} > 2\sigma(G)$ (since G is non-empty, $\sigma(G) > 0$, and therefore F is also non-empty). By the definition of $\sigma(G)$, every subgraph of G must contain a vertex of degree at most $\frac{|E(F)|}{2(|V(F)|-1)}$.

Let $n = |V(F)|$, and enumerate $V(F) = \{v_1, \ldots, v_n\}$ so that for each $i \in [1, n]$, v_i is the vertex of minimum degree in $F_i = F[\{v_1, \ldots, v_i\}]$. Then,

$$\begin{aligned}
|E(F)| &= \sum_{i=1}^{n} \deg_{F_i}(v_i) \\
&= \sum_{i=2}^{n} \deg_{F_i}(v_i) && \text{(since } F_1 = K_1\text{)} \\
&\le \sum_{i=2}^{n} \frac{|E(F)|}{2(n-1)}
\end{aligned}$$

$$
\begin{aligned}
&= (n-1)\frac{|E(F)|}{2(n-1)} \\
&= \frac{|E(F)|}{2},
\end{aligned}
$$

a contradiction since F was non-empty. Therefore for all graphs G, $\Upsilon(G) \leq 2\sigma(G)$.

□

The consequence of Theorem 7.3.2 relevant to this section is that a family of graphs has bounded arboricity iff it has bounded edge density iff it has bounded degeneracy number.

Burr and Erdős admit that Conjecture 7.3.1, if true, is not best possible, as exhibited by the linear Ramsey family of graphs $\mathcal{F} = \{4^i K_i : i \in \mathbb{Z}^+\}$.

Theorem 7.3.3. *The family $\mathcal{F} = \{4^i K_i : i \in \mathbb{Z}^+\}$ has unbounded degeneracy number.*

Proof. For all $i \in \mathbb{Z}^+$,

$$\sigma(4^i K_i) = \max\{\delta(F) : F \subseteq 4^i K_i\} = \delta(4^i K_i) = i - 1.$$

Therefore as i increases, $\sigma(4^i K_i)$ goes to infinity. □

7.4 Arrangeable graphs are linear Ramsey

Chen and Schelp [23] generalized Theorem 7.2.7 with the following (new) family of graphs: for any $k \in \mathbb{Z}^+$, a graph G is *k-arrangeable* iff there exists an ordering $v_1, \ldots, v_n$ of $V(G)$ with the property that for all $i \in [1, n-1]$, if $L_i = G[\{v_1, \ldots, v_i\}]$ and $R_i = G[\{v_i, \ldots, v_n\}]$, then

$$\left| \bigcup \{N_{L_i}(x) : x \in N_{R_i}(v_i)\} \right| \leq k.$$

Stated another way, G is k-arrangeable if there exists an ordering $v_1, \ldots, v_n$ of $V(G)$ with the property that for all $i \in [1, n-1]$,

$$|\{v_j : j \leq i \text{ and } \exists s > i \text{ s.t. } v_s \in N(v_i) \cap N(v_j)\}| \leq k.$$

In order to aid in understanding of what a k-arrangeable graph is, some examples are provided here.

Proposition 7.4.1. *Every cycle is* 2*-arrangeable, and not* 1*-arrangeable.*

Proof. Let $n \in \mathbb{Z}^+$, and let $G = C_n$. To see that G is 2-arrangeable, order $V(G) = \{v_1, v_2, \ldots, v_n\}$ such that $E(G) = \{\{v_i, v_{i+1}\} : i \in [1, n-1]\} \cup \{\{v_1, v_n\}\}$. For each $i \in [1, n-1]$, let $S(i)$ be the statement that

$$\left| \bigcup \{N_{L_i}(x) : x \in N_{R_i}(v_i)\} \right| \leq 2.$$

It suffices to show that for each $i \in [1, n-1]$, $S(i)$ holds. Since $L_1 = K_1$, $S(1)$ holds.

For each $i \in [2, n-1]$,

$$\left|\bigcup\{N_{L_i}(x) : x \in N_{R_i}(v_i)\}\right| = |N_{L_i}(v_{i+1})| \le 2,$$

and therefore $S(i)$ holds, proving G is 2-arrangeable.

To see that G is not 1-arrangeable, let $V(G) = \{v_1, v_2, \ldots, v_n\}$ be any arbitrary ordering of $V(G)$. Set $N_G(v_n) = \{v_i, v_j\}$ with $i < j$. Then

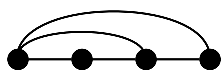
$$\left|\bigcup\{N_{L_i}(x) : x \in N_{R_i}(v_j)\}\right| \ge |N_{L_i}(v_n)| = 2. \qquad \square$$

Note that the first part of the proof of Proposition 7.4.1 relies only on the fact that cycles are 2-regular. Similarly, the second part only relies on the fact that $\delta(G) \ge 2$. Therefore the same arguments can be improved to yield the following corollaries:

Corollary 7.4.2. *Every 2-regular graph is* 2*-arrangeable (and not* 1*-arrangeable).*

Corollary 7.4.3. *Let $k \in \mathbb{Z}^+$, and let G be a graph with $\delta(G) \ge k$. Then G is not $(k-1)$-arrangeable.*

Proposition 7.4.1 (and Corollary 7.4.2) do not exhibit all 2-arrangeable graphs. The graph $G = K_4 \setminus e$ is 2-arrangeable, as shown by the following ordering of the vertices of G.



The following theorems relate "arrangeability" to the degeneracy number and the maximum degree of a graph.

Theorem 7.4.4 (Chen and Schelp, 1993 [23]). *For all $k \in \mathbb{Z}^+$, every k-arrangeable graph is k-degenerate.*

Proof. Let $k \in \mathbb{Z}^+$, and fix a k-arrangeable graph G. For any $i \in [1, n-1]$, let $L_i = G[\{v_1, \ldots, v_i\}]$ and $R_i = G[\{v_i, \ldots, v_n\}]$. Let G be ordered as $V(G) = \{v_1, \ldots, v_n\}$ such that for all $i \in [1, n-1]$,

$$\left| \bigcup \{N_{L_i}(x) : x \in N_{R_i}(v_i)\} \right| \leq k.$$

Let $F \subseteq G$, and let $V(F) = \{w_1, \ldots, w_m\}$ be the ordering of the vertices of F respecting the ordering of $V(G)$. If $\deg_F(w_m) = 0$, then $\delta(F) = 0$. Otherwise, let $w_j \in N_F(w_m)$ be the neighbour of w_m with the highest index with respect to the ordering of $V(G)$ (*i.e.* for all $\ell \in \mathbb{Z}^+$, $j < \ell < m$, $w_\ell \notin N_F(w_m)$). Then,

$$\begin{aligned} \delta(F) &\leq \deg_F(w_m) \\ &= |N_{L_j}(w_m)| \\ &\leq \left| \bigcup \{N_{L_j}(x) : x \in N_{R_j}(w_j)\} \right| \\ &\leq k. \end{aligned}$$

Then since F was an arbitrary subgraph of G, G is k-degenerate. □

Theorem 7.4.5 (Chen and Schelp, 1993 [23]). *For all $d \in \mathbb{Z}^+$, every graph with max degree at most d is $(d(d-1)+1)$-arrangeable.*

Proof. Let G be a graph such that $\Delta(G) \leq d$, and let $v_1, \ldots, v_n$ be an arbitrary ordering of $V(G)$. Then for all $i \in [1, n]$, $|N(v_i)| \leq \Delta(G) \leq d$. Let $L_i = G[\{v_1, \ldots, v_i\}]$

and $R_i = G[\{v_i, \ldots, v_n\}]$. Then $|N_{R_i}(v_i)| \leq d$, and so $|\bigcup\{N_{L_i}(x) : x \in N_{R_i}(v_i)\}| \leq d(d-1)+1$. $\square$

The following theorem is the main result in this section.

Theorem 7.4.6 (Chen and Schelp, 1993 [23]). *For any $k \in \mathbb{Z}^+$, the family of k-arrangeable graphs is linear Ramsey.*

The proof of Theorem 7.4.6 is omitted.

Recall that a graph is *planar* if it can be drawn in the plane with no edges crossing, and *outerplanar* if it can be drawn with all vertices on a circle with no edges crossing. Chen and Schelp showed that every forest is 1-arrangeable (a consequence of Proposition 7.4.1), every outerplanar graph is 3-arrangeable, and every planar graph is 761-arrangeable, thereby proving that the family of planar graphs (which contains all forests and outerplanar graphs) is linear Ramsey. They also show that Theorem 7.4.6 is not equivalent to Conjecture 7.3.1 by constructing an infinite family of graphs with bounded edge density, but for which there is no finite k such that all are k-arrangeable.

7.5 d-degenerate graphs are quadratic Ramsey

In 2003 and 2004, two "major" theorems were proved, making progress towards proving Conjecture 7.3.1. The first, due to Kostochka and Sudakov [89], proved with the probabilistic method that for any $d \in \mathbb{Z}^+$, and any d-degenerate graph G,

$$R(G, G) \leq |V(G)|^{1+o(1)}.$$

The second, due to Kostochka and Rödl [87], showed that if d-degenerate graphs are not linear Ramsey, they are at worst "polynomial" Ramsey (in fact "quadratic" Ramsey). The former is omitted, but the latter is presented here in detail. All following theorems and lemmas, unless otherwise noted, are from [87]. The theorem to be proved is as follows.

Theorem 7.5.1 (Kostochka and Rödl, 2004 [87])**.** *For any $d \in \mathbb{Z}^+$, and for any d-degenerate graph G, there exists a constant $c = c(d)$ such that*

$$R(G, G) \leq c|V(G)|^2.$$

The proof of Theorem 7.5.1 is long (approximately the next sixteen pages), and this result is not used again. Therefore, the proof can be safely skipped by the reader.

The following elementary fact is used (without proof):

Proposition 7.5.2. *Let X be a set, and let $A \subseteq X, B \subseteq X$. Then $|A \cap B| = |A| + |B| - |A \cup B|$, and more specifically, $|A \cap B| \geq |A| + |B| - |X|$.*

Theorem 7.5.3. *Let $k, d \in \mathbb{Z}^+$, and let G be a graph with $\delta(G) \geq k$. Then for any $v_1, \ldots, v_d \in V(G)$,*

$$\left| \bigcap_{i=1}^{d} N(v_i) \right| \geq dk - (d-1)|V(G)|.$$

Proof. The proof is by induction on d. For any $d \in \mathbb{Z}^+$, let $S(d)$ be the statement that for all d vertices $v_1, \ldots, v_d \in V(G)$,

$$\left| \bigcap_{i=1}^{d} N(v_i) \right| \geq dk - (d-1)|V(G)|.$$

Base Case: The statement $S(1)$ reduces to $\forall v \in V(G)$, $|N(v)| \geq k$, which is true.

Inductive Step: Let $m \geq 1$, and assume that $S(m)$ holds. Let $v_1, \ldots, v_{m+1} \in V(G)$. Then

$$\begin{aligned}
\left| \bigcap_{i=1}^{m+1} N(v_i) \right| &\geq \left| \bigcap_{i=1}^{m} N(v_i) \right| + |N(v_{k+1})| - |V(G)| \qquad \text{(by Proposition 7.5.2)} \\
&\geq mk - (m-1)|V(G)| + |N(v_{k+1})| - |V(G)| \qquad \text{(since } S(m) \text{ holds)} \\
&\geq mk - (m-1)|V(G)| + k - |V(G)| \qquad \text{(since } \delta(G) \geq k\text{)} \\
&= (m+1)k - m|V(G)|.
\end{aligned}$$

Therefore $S(m+1)$ holds, and by induction, for all $d \in \mathbb{Z}^+$, $S(d)$ holds. □

Lemma 7.5.4. *Let $c \geq 0, \lambda \geq 0, n \geq 1$, and let H be a graph on n vertices with $E(H) \geq (c+\lambda)\binom{n}{2}$. Then there exists an induced subgraph $H' \preceq H$ such that*

$$\delta(H') \geq c(|V(H')| - 1) + \frac{\lambda n}{2}.$$

Proof. Assume the lemma is not true. Then every induced subgraph H' of H has

$$\delta(H') < c(|V(H')| - 1) + \frac{\lambda n}{2}.$$

Sort the vertices of H as $v_1, \ldots, v_n$ such that in the graph $H_i = H[\{v_1, \ldots, v_i\}]$, the vertex v_i is of minimum degree. Then $K_1 = H_1 \preceq H_2 \preceq \ldots \preceq H_n = H$.

Counting the edges of H,

$$\begin{aligned}
|E(H)| &= \sum_{i=2}^{n} \deg_{H_i}(v_i) \\
&< \sum_{i=2}^{n} \left(c(|V(H_i)| - 1) + \frac{\lambda n}{2} \right) \qquad \text{(by assumption)} \\
&= c \sum_{i=2}^{n} (i-1) + \frac{\lambda n}{2} \sum_{i=2}^{n} 1 \\
&= c \sum_{i=1}^{n-1} (i) + \frac{\lambda n(n-1)}{2} \\
&= c\binom{n}{2} + \lambda \binom{n}{2} \\
&\leq |E(H)| \qquad \text{(by the definition of } H\text{).}
\end{aligned}$$

Thus $|E(H)| < |E(H)|$, a contradiction. $\square$

The proof of the main result, Theorem 7.5.1, is separated into two cases. First, it is shown that if a graph H is large enough, and contains a large subgraph without too many edges, then $\overline{H}$ contains every d-degenerate graph. Second, it is shown that if a graph H is large enough, and every large subgraph contains many edges, then H must contain every d-degenerate graph.

Let $d, n, s \in \mathbb{Z}^+$. A graph is said to have the (d,n)-*property* iff every d vertices have at least $n-d$ common neighbours, *i.e.*, G has the (d,n)-property iff for all distinct d vertices $v_1, \dots, v_d \in V(G)$,

$$|N(v_1) \cap \cdots \cap N(v_d)| \geq n - d.$$

For example, the only graph on n vertices with the $(1,n)$-property is K_n.

Lemma 7.5.5. *Let $n, d \in \mathbb{Z}^+$, and let H be a graph with $|V(H)| > 4n$ such that there exists an induced subgraph $H' \preceq H$ on at least $4n$ vertices satisfying*

$$|E(H')| < \frac{1}{2d}\binom{|V(H')|}{2}.$$

Then $\overline{H'}$ contains a subgraph with the (d,n)-property.

Proof. Let $k = |V(H')|$. Then,

$$\begin{aligned} |E(\overline{H'})| &\geq \binom{k}{2} - \left[\frac{1}{2d}\binom{k}{2}\right] \\ &= \left(1 - \frac{1}{2d}\right)\binom{k}{2} \\ &= \left(1 - \frac{1}{d} + \frac{1}{2d}\right)\binom{k}{2}. \end{aligned}$$

Let $c = 1 - \frac{1}{d}$, $\lambda = \frac{1}{2d}$. Then by Lemma 7.5.4, there exists $\overline{H_1} \subseteq \overline{H'}$ such that

$$\begin{aligned} \delta(\overline{H_1}) &\geq c(|V(\overline{H_1})| - 1) + \lambda\frac{|V(\overline{H'})|}{2} \\ &= \left(1 - \frac{1}{d}\right)(|V(H_1)| - 1) + \frac{1}{2d}\left(\frac{k}{2}\right) \\ &= \frac{d-1}{d}(|V(H_1)| - 1) + \frac{k}{4d}. \end{aligned}$$

Then for all $v_1, \ldots, v_d \in V(H_1)$,

$$\begin{aligned}
\left|\bigcap_{i=1}^{d} N_{\overline{H_1}}(v_i)\right| &\geq d\left(\frac{d-1}{d}(|V(H_1)|-1)+\frac{k}{4d}\right)-(d-1)|V(H_1)| \quad \text{(by Thm 7.5.3)}\\
&= (d-1)(|V(H_1)|-1)+\frac{k}{4}-(d-1)|V(H_1)|\\
&= -(d-1)+\frac{k}{4}\\
&\geq -d+1+\frac{4n}{4} \qquad \text{(since } |V(H')| \geq 4n)\\
&= n-d+1\\
&> n-d.
\end{aligned}$$

Therefore $\overline{H_1} \subseteq \overline{H'}$ has the (d,n)-property. □

Lemma 7.5.6. *For all $d, n \in \mathbb{Z}^+$, every graph with the (d,n)-property contains as a subgraph every d-degenerate graph on n vertices.*

Proof. Let $d, n \in \mathbb{Z}^+$, let H be a graph with the (d,n)-property, and let G be a d-degenerate graph on n vertices. Order the vertices of G as $x_1, \ldots, x_n$ such that x_i is of minimum degree in $G_i = G[\{x_1, \ldots, x_i\}]$. For any $m \in [1,n]$, let $J(m)$ be the statement that there exists a function f embedding G_m into H.

Base Case: Since $G_1 = K_1$, $J(1)$ holds.

Inductive Step: Let $k \in [1, n-1]$, and assume that $J(k)$ holds. Let $f : \{x_1, \ldots, x_k\} \to V(H)$ be an embedding of G_k into H, and let $Y = N_{G_{k+1}}(x_{k+1})$.

Informally, let $S \in [V(H)]^d$ be such that $f(Y) \subseteq S$, and

$$|N_H(S) \cap (V(H) \setminus f(G_k))| \geq 1.$$

This unusual definition for S is due to the fact that S always needs to be a set of d vertices, should contain $f(Y)$, and needs to be such that there is at least one common neighbour of the vertices of S outside of $f(G_k)$. Formally, if $k < d$, let $S \in [V(H)]^d$ be any set such that $f(G_k) \subseteq S$. Otherwise, if $k \geq d$, let $S \in [V(H)]^d$ be any set such that $f(Y) \subseteq S \subseteq f(G_k)$ (see Figure 7.2). Since $|S| = d$, and H has

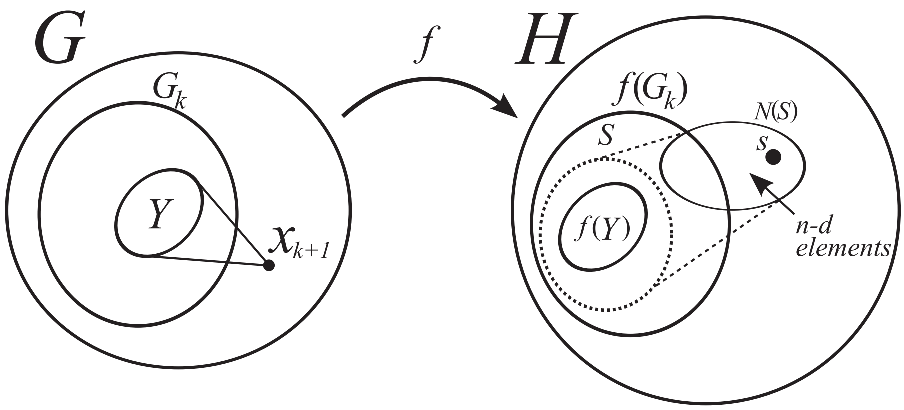


Figure 7.2: The embedding f, and the set S.

the (d, n)-property, the vertices in S have $n-d$ common neighbours in H, and since

$$|f(G_k) \setminus S| = k - d \leq n - 1 - d < n - d,$$

at least one common neighbour of the vertices of S, call it s, is outside $f(G_k)$. Extend f to f^* by defining $f^*(x_{k+1}) = s$. Then f^* is an embedding of G_{k+1} into H, and therefore $J(k+1)$ holds.

Thus by induction, for all $m \in \mathbb{Z}^+$, $J(m)$ holds, and Lemma 7.5.6 is proved. $\square$

A graph H is said to be (d,s)-*thick* if for every induced subgraph $H' \preceq H$ on at least s vertices,

$$|E(H')| \geq \frac{1}{2d}\binom{|V(H')|}{2}.$$

In other words, every "big" subgraph has "many" edges. Therefore if a graph is not (d,s) thick, then there exists a large subgraph with not too many edges.

Lemma 7.5.7. *Let $d,n \in \mathbb{Z}^+$, let G be any d-degenerate graph on n vertices, and let H be a graph on more than $4dn$ vertices that is* <u>not</u> *$(d,4dn)$-thick. Then $G \subseteq \overline{H}$.*

Proof. Since H is not $(d,4dn)$-thick, there exists an $H' \preceq H$ on at least $4dn$ vertices such that

$$|E(H')| < \frac{1}{2d}\binom{|V(H')|}{2}.$$

Then by Lemma 7.5.5, $\overline{H'}$ contains a subgraph with the (d,n)-property, which by Lemma 7.5.6, contains G. $\square$

Therefore every graph that is not $(d,4dn)$-thick contains a copy of G in its complement. Next it is proved that every large enough $(d,4dn)$-thick graph contains G. To begin with, the following elementary lemma is required.

Lemma 7.5.8. *Let $k \in \mathbb{Z}^+$, and let X be a set. Let $B \subseteq X$ such that for every $A \in [X]^k$, $A \cap B \neq \emptyset$. Then $|B| \geq |X| - (k-1)$.*

Proof. If not, then $B \leq |X| - k$. However, then $|X \setminus B| = |X| - |B| \geq |X| - (|X| - k) = k$, and any set $A \in [X \setminus B]^k$ is such that $A \cap B = \emptyset$, a contradiction. $\square$

Let $r, m, d \in \mathbb{Z}^+$. For a graph H, an (H, r, m, d)-*reducing pair* is defined to be a pair of sets (R, S) such that $R, S \subseteq V(H)$, $R \cap S = \emptyset$, $|R| = r$, $|S| \geq \frac{3}{4}\frac{|V(H)|}{m^{d-1}}$, and for all $v \in R$, $|N_H(v) \cap S| \leq \frac{4}{3}\frac{|S|}{m}$.

Lemma 7.5.9. *Let $d \geq 2, r \geq 2, m \geq 8d$, and let H be a graph with at least $2rm^{4d^2-d}$ vertices. If every subgraph $H_1 \subseteq H$ with $|V(H_1)| > \frac{1}{m^{4d^2}}|V(H)|$ contains an (H_1, r, m, d)-reducing pair, then H contains a subgraph H' on exactly $4dr$ vertices with $|E(H')| < \frac{1}{2d}\binom{4dr}{2}$. Consequently, H is not $(d, 4dr)$-thick.*

Proof. Assume that every subgraph $H_1 \subseteq H$ with $|V(H_1)| > \frac{1}{m^{4d^2}}|V(H)|$ contains an (H_1, r, m, d)-reducing pair. Construct a sequence of graphs

$$H_0 \succ H_1 \succ \cdots \succ H_{4d-1}$$

in the following inductive way: Let $H_0 = H$. Let $k \in [0, 4d-2]$, and assume that H_k has been defined such that $|V(H_k)| > \frac{1}{m^{kd}}|V(H)|$. By assumption, since

$$\frac{1}{m^{kd}} > \frac{1}{m^{4d^2}},$$

an (H_k, r, m, d)-reducing pair (R_k, S_k) can be chosen in H_k. The number of edges between R_k and S_k, $|E(R_k, S_k)|$, is

$$|E(R_k, S_k)| = \sum_{v \in R_k} |N_{H_k}(v) \cap S_k|$$

$$\begin{aligned} &\leq \sum_{v\in R_k} \frac{4}{3}\frac{|S_k|}{m} && \text{(by the definition of a reducing pair)} \\ &= |R_k|\frac{4}{3}\frac{|S_k|}{m}. && (7.3) \end{aligned}$$

Claim. There exists an $S'_k \subseteq S_k$ with $|S'_k| \geq \frac{|S_k|}{3}$ such that for all $v \in S'_k$,

$$|N_{H_k}(v) \cap R_k| \leq \frac{2}{m}|R_k|.$$

Proof of claim. Suppose the claim is not true, *i.e.*, for all $S'_k \subseteq S_k$ with $|S'_k| \geq \frac{|S_k|}{3}$, there exists $v \in S'_k$ such that

$$|N_{H_k}(v) \cap R_k| > \frac{2}{m}|R_k|.$$

Then, by Lemma 7.5.8, there are at least

$$1 + \left(|S_k| - \left\lceil\frac{|S_k|}{3}\right\rceil\right) > \frac{2}{3}|S_k|$$

vertices v in S_k such that $|N_{H_k}(v) \cap R_k| > \frac{2}{m}|R_k|$, and therefore $|E(R_k, S_k)|$ counted a second way is

$$\begin{aligned} |E(R_k, S_k)| &= \sum_{v\in S_k} |N_{H_k}(v) \cap R_k| \\ &> \frac{2}{3}|S_k|\left(\frac{2}{m}|R_k|\right) \\ &= |R_k|\frac{4}{3}\frac{|S_k|}{m} \\ &\geq |E(R_k, S_k)|, && \text{(by equation (7.3))} \end{aligned}$$

which is a contradiction, proving the claim.

Let H_{k+1} be the subgraph of H induced on the vertices S'_k. Then

$$\begin{aligned}
|V(H_{k+1})| &\geq \frac{|S_k|}{3} \\
&\geq \frac{1}{3}\left(\frac{3}{4}\frac{|V(H_k)|}{m^{d-1}}\right) && \text{(by the definition of a reducing pair)} \\
&= \frac{m}{4}\frac{|V(H_k)|}{m^d} \\
&\geq \frac{|V(H_k)|}{m^d} && \text{(since } m \geq 8d \geq 16\text{)} \\
&> \frac{1}{m^d}\frac{|V(H)|}{m^{kd}} && \text{(by induction hypothesis)} \\
&= \frac{|V(H)|}{m^{(k+1)d}}.
\end{aligned}$$

Therefore $|V(H_{k+1})| > \frac{|V(H)|}{m^{(k+1)d}}$, which completes the inductive step.

Notice that

$$|V(H_{4d-1})| = |S'_{4d-2}| \geq \frac{|V(H)|}{m^{(4d-1)d}} \geq \frac{2rm^{4d^2-d}}{m^{4d^2-d}} = 2r > r.$$

Let R_{4d-1} be any subset of S'_{4d-2} of cardinality r. Let $\tilde{R} = \bigcup_{k=0}^{4d-1} R_k$ and $\tilde{H} = H[\tilde{R}]$. Since for all $k = 0, \ldots, 4d-1$, $|R_k| = r$, and they are all disjoint, it follows that $|\tilde{R}| = 4dr$, and since $R_{k+1} \subseteq S'_k$, for all $i < j$,

$$\begin{aligned}
|E_H(R_i, R_j)| &\leq |R_j|\frac{2}{m}|R_i| && \text{(by definition of } S'_k\text{)} \\
&= \frac{2r^2}{m}.
\end{aligned}$$

Thus,

$$|E(\tilde{H})| = \sum_{i=0}^{4d-1} E(H[R_i]) + \sum_{i<j} E_H(R_i, R_j)$$

$$
\begin{aligned}
&\le 4d\binom{r}{2} + \binom{4d}{2}\frac{2r^2}{m} \\
&= (2dr)\left(r - 1 + \frac{2(4d-1)r}{m}\right) \\
&= (2dr)(4dr-1)\left(\frac{r-1}{4dr-1} + \frac{8dr-2r}{m(4dr-1)}\right) \\
&= (2dr)(4dr-1)\left(\frac{1}{4d}\left(\frac{r-1}{r-\frac{1}{4d}}\right) + \frac{2}{m}\left(\frac{4dr-r}{4dr-1}\right)\right) \\
&< (2dr)(4dr-1)\left(\frac{1}{4d} + \frac{2}{m}\right) \\
&\le \binom{4dr}{2}\left(\frac{1}{4d} + \frac{2}{8d}\right) \qquad \text{(since } m \ge 8d\text{)} \\
&= \frac{1}{2d}\binom{4dr}{2}.
\end{aligned}
$$

Therefore $\tilde{H} \subseteq H$ has $4dr$ vertices, and less than $\frac{1}{2d}\binom{4dr}{2}$ edges, proving the lemma.

$\square$

Let H be a graph with $V(H) = M$. A set $A \subseteq V(H)$ is called (H, m)-*good* iff

$$\left|\bigcap_{v \in A} N_H(v)\right| \ge \frac{M}{m^{|A|}},$$

and is (H, m)-*bad* otherwise.

Lemma 7.5.10. *Fix $m \ge 2, r, d$. Let H be a graph such that $|V(H)| \ge 2rm^d$. Then if there exists an (H, m)-good set A in H of order at most $d - 1$, and there exist r elements $v_1, \ldots, v_r$ such that for all $i \in [1, r]$, $A \cup \{v_i\}$ is (H, m)-bad, then H contains an (H, r, m, d)-reducing pair. Consequently, if H contains no (H, r, m, d)-reducing pair, then there are at most $r - 1$ (H, m)-bad singletons.*

Proof. Fix a graph H with $|V(H)| \geq 2rm^d$ that contains a (H, m)-good set A, and r elements $v_1, \ldots, v_r$ such that for all $i \in [1, r]$, $A \cup \{v_i\}$ is (H, m)-bad. Let $R = \{v_1, \ldots, v_r\}$ and let $S = N_H(A) \setminus R$ (see Figure 7.3).

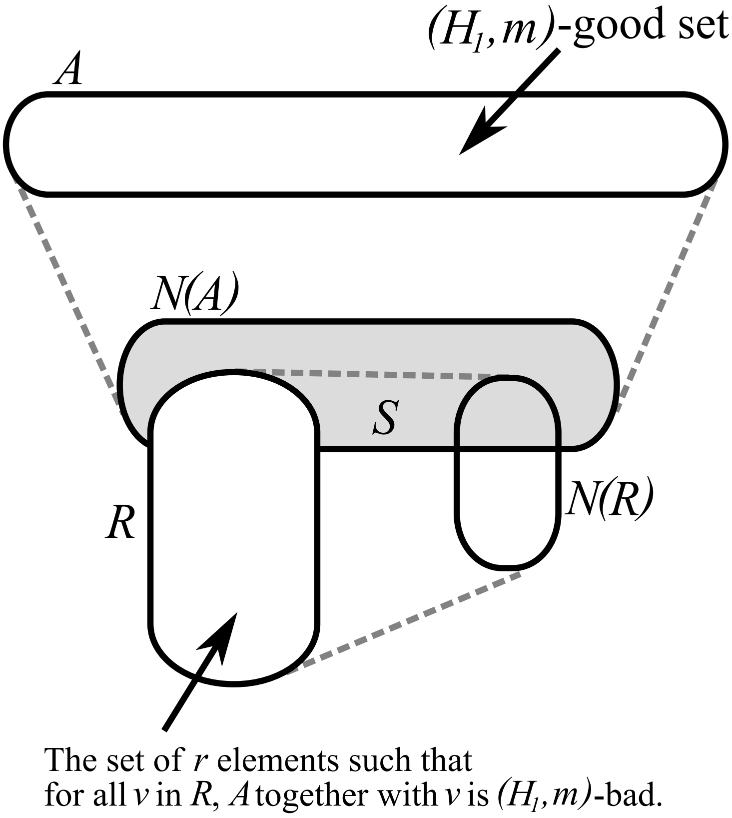


Figure 7.3: Construction of S and R

At this point we locally define $N_H(\emptyset) = V(H)$, in order for the proof to work when $A = \emptyset$. The cardinality of S is

$$\begin{aligned} |S| &= |N_H(A) \setminus R| \\ &= |N_H(A)| - |N_H(A) \cap R| \end{aligned}$$

$$
\begin{aligned}
&\geq |N_H(A)| - r \\
&\geq \frac{|V(H)|}{m^{|A|}} - r && \text{(since } A \text{ is } (H,m)\text{-good)} \\
&\geq \frac{|V(H)|}{m^{|A|}} - \frac{|V(H)|}{2m^d} && \text{(since } |V(H)| \geq 2rm^d\text{)} \\
&= \frac{|V(H)|}{m^{|A|}}\left(1 - \frac{1}{2m^{d-|A|}}\right) \\
&\geq \frac{|V(H)|}{m^{|A|}}\left(1 - \frac{1}{4m^{d-1-|A|}}\right) && \text{(since } m \geq 2\text{)} \\
&\geq \frac{|V(H)|}{m^{|A|}}\left(1 - \frac{1}{4}\right) && \text{(since } |A| \leq d-1\text{)} \\
&= \frac{3}{4}\frac{|V(H)|}{m^{|A|}} && (7.4) \\
&\geq \frac{3}{4}\frac{|V(H)|}{m^{d-1}} && \text{(again since } |A| \leq d-1\text{).}
\end{aligned}
$$

Further, for all $v \in R$,

$$
\begin{aligned}
|N_H(v) \cap S| &\leq |N_H(R) \cap N_H(A)| && \text{(since } N_H(v) \cap S \subseteq N_H(R) \cap N_H(A)\text{)} \\
&\leq |N_H(v_1) \cap N_H(A)| \\
&< \frac{1}{m}\frac{|V(H_1)|}{m^{|A|}} && \text{(since } A \cup \{v_1\} \text{ is } (H,m)\text{-bad)} \\
&\leq \frac{1}{m}\frac{4}{3}|S| && \text{(by equation (7.4))} \\
&= \frac{4}{3}\frac{|S|}{m}.
\end{aligned}
$$

Therefore (R, S) is an (H, r, m, d)-reducing pair. The second part of the lemma is shown by noticing that $A = \emptyset$ is an (H, m)-good set, and using the contrapositive of the first part. $\square$

Lemma 7.5.11. *Fix $n, m, \Delta, d, \alpha \in \mathbb{R}^+$ such that $n > \Delta \geq d \geq 2$, $m \geq d$, and*

$\alpha \geq 1$. *Let* $M_0 = m^d \Delta \alpha n$. *If a graph* H *on* $M_1 > M_0$ *vertices has no* $(H, \alpha n, m, d)$-*reducing pairs, then every* d-*degenerate graph* G *on* n *vertices with maximum degree* Δ *can be embedded into* H.

Proof. Order the vertices of G as $x_1, \ldots, x_n$ such that for all $i = 1, \ldots, n$, and $G_i = G[\{x_1, \ldots, x_i\}]$, $deg_{G_i}(x_i) \leq d$. An embedding f of G_i into H is inductively defined, maintaining the property that

$$\forall j \in [i+1, n], \qquad f(N_{G_j}(x_j) \cap \{x_1, \ldots, x_i\}) \text{ is } (H, m)\text{-good.} \tag{7.5}$$

Since H has no $(H, \alpha n, m, d)$-reducing pair, the consequence of Lemma 7.5.10 says that there are at most $\alpha n - 1$ (H, m)-bad singletons. Since $|V(H)| > m^d \Delta \alpha n > \alpha n$, there exists an element, call it v_1, such that $\{v_1\}$ is (H, m)-good. Therefore define $f(x_1) = v_1$.

Let $k \geq 1$, and assume that f embeds G_k into H, maintaining property (7.5). Enumerate

$$N_{G_k}(x_k) = \{x_{i_1}, x_{i_2}, \ldots, x_{i_{d'}}\},$$

and let $A = f(N_{G_k}(x_k))$. It remains to embed x_{k+1} into $N_H(A)$, maintaining the property that

$$\forall j = k+2, \ldots, n \qquad f(N_{G_j}(x_j) \cap \{x_1, \ldots, x_k, x_{k+1}\}) \text{ is } (H, m)\text{-good.}$$

Let $j \in [k+2, \ldots, n]$. If $x_{k+1} \notin N_{G_j}(x_j)$, then $f(N_{G_j}(x_j) \cap \{x_1, \ldots, x_{k+1}\})$ is

(H, m)-good since f satisfies property (7.5), so assume $x_{k+1} \in N_{G_j}(x_j)$.

Since f satisfies property (7.5), $f(N_{G_j}(x_j) \cap \{x_1, \ldots, x_k\})$ is also (H, m)-good. Since $x_{k+1} \in N_{G_j}(x_j)$,

$$f(N_{G_j}(x_j) \cap \{x_1, \ldots, x_{k+1}\}) = f(N_{G_j}(x_j) \cap \{x_1, \ldots, x_k\}) \cup \{f(x_{k+1})\} \tag{7.6}$$

Then by Lemma 7.5.10, there are at most $\alpha n - 1$ elements that f cannot map x_{k+1} to. Therefore for each $j \in [k+2, n]$ with $x_{k+1} \in N_{G_j}(x_j)$, there are at most $\alpha n - 1$ elements $f(x_{k+1})$ cannot be. Consider two cases:

Case 1. If $A = \emptyset$, then

$$\begin{aligned}
&|N_H(A) \setminus f(G_k)| - |N_G(x_{k+1}) \setminus G_k|(\alpha n - 1) \\
&= M_1 - k - |N_G(x_{k+1}) \setminus G_k|(\alpha n - 1) \\
&\geq M_1 - k - \Delta(\alpha n - 1) \\
&= M_1 - \alpha n - \Delta\alpha n + \Delta \quad (\text{since } k \leq \alpha n) \\
&> M_1 - \alpha n(1 + \Delta) \\
&> M_0 - \alpha n(1 + \Delta) \\
&> 0.
\end{aligned}$$

Case 2. If $A \neq \emptyset$, then

$$|N_H(A) \setminus f(G_k)| - |N_G(x_{k+1}) \setminus G_k|(\alpha n - 1)$$

$$\begin{aligned}
&\geq |N_H(A)| - k - (\Delta - 1)(\alpha n - 1) \\
&= |N_H(A)| - \Delta\alpha n + (\Delta + \alpha n - (k+1)) \\
&\geq \frac{|V(H)|}{m^{|A|}} - \Delta\alpha n \\
&\geq \frac{|V(H)|}{m^d} - \Delta\alpha n \\
&> \frac{m^d \Delta\alpha n}{m^d} - \Delta\alpha n \\
&= 0.
\end{aligned}$$

Therefore, x_{k+1} can be mapped to at least one element without causing any new bad sets. The inductive embedding is now complete, and the lemma is proved. □

Theorem 7.5.12. *Let $d, n \in \mathbb{Z}^+$, and let G be a d-degenerate graph on n vertices with maximum degree Δ. Let H be an arbitrary graph with at least $(8d)^{4d^2+d}\Delta n$ vertices. Then either $G \subseteq H$, or $G \subseteq \overline{H}$.*

Proof. If H is not $(d, 4dn)$-thick, then by Lemma 7.5.7, $G \subseteq \overline{H}$, and the proof is done. If H is $(d, 4dn)$-thick, then by the contrapositive of Lemma 7.5.9 (with $m = 8d$ and $r = n$), there exists an $H_1 \subseteq H$ with

$$|V(H_1)| > \frac{1}{(8d)^{4d^2}}|V(H)| \geq (8d)^d \Delta n$$

that does not contain any $(H_1, n, 8d, d)$-reducing pairs. Then by Lemma 7.5.11 (with $H = H_1$, $\alpha = 1$ and $m = 8d$), $G \subseteq H_1$. This proves the theorem. □

The following corollary proves Theorem 7.5.1.

Corollary 7.5.13. *For any $d \in \mathbb{Z}^+$, and any d-degenerate graph G,*

$$R(G,G) \leq (8d)^{4d^2+d}\Delta(G)|V(G)|.$$

7.6 Ramsey-size linear graphs

Recently another type of linear Ramsey has been defined, where "linear" instead refers to linear in the number of edges.

By Theorem 4.4.2, for any tree T, $R(K_3, T) = 2|E(T)|+1$. In 1980, Harary [6, 131] asked if it was true that for every graph G, $R(K_3, G) \leq 2|E(G)| + 1$. Sidorenko [131] proved this bound in 1993, and this motivated the following definition: Let G be a graph with at least three vertices. Then G is defined to be *Ramsey size linear* iff there exists a constant $c = c(G)$ such that for every graph H with no isolated vertices, $R(G, H) \leq c|E(H)|$.

Theorem 7.6.1 (Erdős, Faudree, Rousseau, and Schelp, 1993 [39])**.** *Let G be a connected graph. If $|E(G)| \leq |V(G)|+1$ then G is Ramsey size linear. If $|E(G)| \geq 2(|V(G)|-1)$, then G is not Ramsey size linear. Further, for all $n \in \mathbb{Z}^+$, and for all $e \in [n+2, 2n-3]$, there exist graphs G_1 and G_2 each on n vertices and e edges, one of which is Ramsey size linear, and one of which is not.*

Note that due to Theorem 7.6.1, K_4 is not Ramsey size linear. It is interesting to note that this fact can be verified using Spencer's [133] bound on $R(K_4, K_t)$ (Theorem 3.4.9):

$$R(K_4, K_t) > c\left(\frac{t}{\log t}\right)^{\frac{5}{2}}.$$

It can be shown that, for large enough t, $\left(\frac{t}{\log t}\right)^{\frac{5}{2}} > \binom{t}{2}$. Therefore, there can be no constant c such that for every $t \in \mathbb{Z}^+$, $R(K_4, K_t) < c\binom{t}{2}$, which verifies that K_4 is not Ramsey size linear.

Chapter 8

Restricted Ramsey theorems

Restricted graph Ramsey questions are of the form, for some $r \in \mathbb{Z}^+$ and some graphs $G_1, \ldots, G_r$, does there exists a graph F so that $F \longrightarrow (G_1, \ldots, G_r)_r^{K_2}$, and such that F satisfies some restrictive property (like F is K_4-free, or 2-connected)? This chapter presents some representative theorems answering restricted graph Ramsey questions.

8.1 Graphs arrowing K_3

Given a family of graphs $\mathcal{G}$, define $\mathrm{Forb}(\mathcal{G})$ as the class of graphs which do not contain induced subgraphs isomorphic to any element of $\mathcal{G}$. If $\mathcal{G} = \{G\}$, a family consisting of a single graph G, then write Forb(G) rather than Forb($\{G\}$). Also, say $F \in$ Forb(G) is *G-free*. In Section 3.1.2, it was shown that $K_6 \longrightarrow (K_3)_2^{K_2}$. It follows that any graph that contains a K_6 also arrows K_3. Is there a K_6-free graph (*i.e.*, a graph in Forb(K_6)) that arrows $(K_3)_2^{K_2}$? Erdős and Hajnal asked this question in 1967 [41], and it was answered by Graham the following year:

Theorem 8.1.1 (Graham, 1968 [61]). *$K_3 + C_5 \longrightarrow (K_3)_2^{K_2}$, and no other K_6-free graph on fewer than eight vertices has this property.*

The proof is done by checking cases, similar to the proof of $K_6 \longrightarrow (K_3)_2^{K_2}$, but longer, and is omitted. One can verify (see Figure 8.1) that K_3+C_5 does not contain

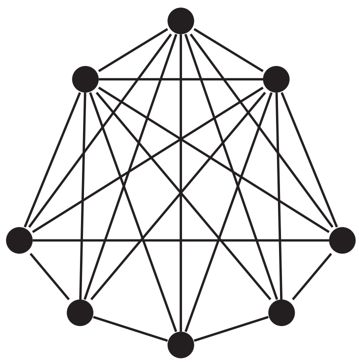


Figure 8.1: The graph $K_3 + C_5$

a K_6, but it does contain a number of K_5's. Two years later, Folkman published the following theorem:

Theorem 8.1.2 (Folkman, 1970 [53]). *For all $a \in \mathbb{Z}^+$, there exists a graph $F \in$ Forb(K_{a+1}) such that $F \longrightarrow (K_a)_2^{K_2}$.*

Theorem 8.1.2 is (trivially) best possible in terms of forbidding complete graphs. Folkman's proof of Theorem 8.1.2 was constructive, but the graph constructed by Folkman's proof was astronomically large. To illustrate how large it was, Erdős (see [135]) later offered a reward for producing such a graph with less than 10^{10} vertices. Erdős was given an affirmative answer by Spencer in 1988 [135] (with an erratum a year later [136]). Spencer showed that there is positive probability that a random graph on $3 \cdot 10^9$ vertices with a random edge removed from each copy of K_4 will arrow $(K_3, K_3)_2^{K_2}$. According to [135], Spencer's proof technique is so specific that it only works for arrowing $(K_3, K_3)_2^{K_2}$.

8.2 Folkman Numbers

For any $r, a, m_1, \ldots, m_r \in \mathbb{Z}^+$, define the *vertex-colouring Folkman number* as

$$F_v(m_1, \ldots, m_r; a) = \min\{|V(F)| : F \in \text{Forb}(K_a), F \longrightarrow (K_{m_1}, \ldots, K_{m_r})_r^{K_1}\},$$

and define the *edge-colouring Folkman number* as

$$F_e(m_1, \ldots, m_r; a) = \min\{|V(F)| : F \in \text{Forb}(K_a), F \longrightarrow (K_{m_1}, \ldots, K_{m_r})_r^{K_2}\}.$$

Very few exact values of the Folkman numbers are known. The solution to the party problem shows that $F_e(3, 3; 7) = 6$. By Theorem 8.1.1, $F_e(3, 3; 6) = 8$. Note that if $a < b$, then $F_e(m_1, \ldots, m_r; a) \geq F_e(m_1, \ldots, m_r; b)$ since forbidding K_a also forbids

K_b. Also, if $a > R(m_1, \ldots, m_r)$, then $F_e(m_1, \ldots, m_r; a) = R(m_1, \ldots, m_r)$.

In 1999, Piwakowski, Radziszowski, and Urbański [119] showed $F_e(3, 3; 5) = 15$. Radziszowski and Xiaodong [125] recently proved that $F_e(3, 3; 4) \geq 19$, and this result, together with Spencer's upper bound are currently the best known bounds for $F_e(3, 3; 4)$:

$$19 \leq F_e(3, 3; 4) \leq 3 \cdot 10^9.$$

Radziszowski and Xiaodong [125] conjecture that there may be a graph with as few as 127 vertices that is K_4-free, and arrows $(K_3)_2^{K_2}$, and that $F_e(3, 3; 4)$ may be even less than 100.

In terms of the vertex-colouring Folkman numbers, for $r, a, m_1, \ldots, m_r \in \mathbb{Z}^+$, and $M = 1 + \sum_{i=1}^r m_i$, it follows by the pigeonhole principle that $R_1(m_1, \ldots, m_r) = M$, and therefore if $a > M$, $F_v(m_1, \ldots, m_r; a) = M$. For the case $a = M$, Łuczak and Urbański [95] showed that

$$F_v(m_1, \ldots, m_r; M) = \max\{m_1, \ldots, m_r\} + M.$$

When $a = M - 2$, only one non-trivial result is known, namely that $F_v(2, 2, 2, 2; 3) = 22$ [77]. According to [30], most of the research done on this topic has been in the case $a = M - 1$. For more known results, see, *e.g.*, [30, 94, 95, 125].

8.3 The Erdős girth-chromatic number theorem

If a graph has high chromatic number, one might expect that this graph has short cycles. However, in 1959, Erdős proved that there are graphs with no small cycles, and yet with high chromatic number. Recall that the *girth* of a graph G is the length of the shortest cycle in G, that is,

$$\text{girth}(G) = \min\{|V(C)| : C \text{ is a cycle in } G\}.$$

Theorem 8.3.1 (Erdős, 1959 [35]). *Given any positive integers $r \geq 2, p \geq 3$, there exists a graph G with girth$(G) > p$ and $\chi(G) > r$.*

The proof of Theorem 8.3.1 is omitted (see, *e.g.*, [3, pp. 38–39] or [79, pp. 265–266]). On the surface, Theorem 8.3.1 does not appear to be a restricted graph Ramsey theorem, but it can be restated in the following "restricted Ramsey-manner":

Theorem (Erdős girth-chromatic number theorem restated). *Given any positive integers $r \geq 2, p \geq 3$, there exists a graph $F \in$ Forb$(\{C_3, \ldots C_p\})$ such that $F \longrightarrow (K_2)_r^{K_1}$.*

The Erdős girth-chromatic number theorem has also been generalized to k-uniform hypergraphs. A (simple) cycle of length n in a hypergraph G is a sequence

$$(v_1, e_1, v_2, e_2, \ldots, v_n, e_n)$$

of distinct alternating vertices and hyperedges such that for each $i \in [1, n-1]$,

$\{v_i, v_{i+1}\} \subseteq e_i$, and $\{v_1, v_n\} \subseteq e_n$. For $i, k \geq 2$, let C_i^k denote a k-uniform hypergraph that is a cycle of length i containing no smaller cycle. For $i \geq 3$, hyperedges in C_i^k intersect in at most one vertex, forcing C_i^k to be unique up to isomorphism. The Erdős girth-chromatic number theorem can then be restated for hypergraphs (see [40]): given any positive integers $k, r, p \geq 2$, there exists a k-uniform hypergraph G with $\text{girth}(G) > p$ and $\chi(G) > r$, and therefore, if e_k denotes an arbitrary k-hyperedge, there exists a k-uniform hypergraph $F \in \text{Forb}(\{C_2^k, \ldots C_p^k\})$ such that $F \longrightarrow (e_k)_r^{K_1}$. To see why this Ramsey translation holds, it suffices to see (trivially) that for any hypergraph F, $\text{girth}(F) > p$ implies that $F \in \text{Forb}(\{C_2^k, \ldots, C_p^k\})$.

Theorem 8.3.1 can be applied to produce a restricted Ramsey theorem about 2-connected graphs. Recall a graph is said to be *n-connected* if it is connected and cannot be made disconnected by removing $n-1$ vertices (*e.g.*, a graph is 2-connected if it cannot be disconnected by removing one vertex).

Theorem 8.3.2 (Nešetřil and Rödl, 1976 [105]). *Let $\mathcal{A}$ be any finite family of 2-connected graphs. Then for every $r \in \mathbb{Z}^+$, $G \in Forb(\mathcal{A})$, there exists an $F \in Forb(\mathcal{A})$ such that*

$$F \xrightarrow{ind} (G)_r^{K_1}.$$

Proof. Let $\mathcal{A}$ be any such finite family, and let $r \in \mathbb{Z}^+$. Let

$$p = \min\{|V(A)| : A \in \mathcal{A}\}.$$

If $p = 1$, Forb($\mathcal{A}$) is empty. If $p = 2$, then $K_2 \in \mathcal{A}$, and therefore Forb($\mathcal{A}$) is made up of empty graphs, and the theorem is proved by the pigeonhole principle.

Assume $p > 2$. Let $G \in \text{Forb}(\mathcal{A})$, let $k = |V(G)|$, and let $q = \max\{|V(A)| : A \in \mathcal{A}\}$. By Theorem 8.3.1, there exists a k-uniform hypergraph H with $\text{girth}(H) > q$ and $\chi(H) > r$. For each $e \in E(H)$, fix a bijection $\psi_e : V(G) \to e$. Let F be the graph with $V(F) = V(H)$, and

$$E(F) = \{\{\psi_e(v_1), \psi_e(v_2)\} : \{v_1, v_2\} \in E(G) \text{ and } e \in E(H)\}$$

(that is, replace each k-edge of H with a copy of G). Note that $E(F)$ is well defined since $\text{girth}(H) > q > p > 2$, and therefore edges of H intersect in at most one vertex.

To see that $F \in \text{Forb}(\mathcal{A})$, let $C \preceq F$ be any 2-connected subgraph of F. If there exists $e \in E(H)$ such that $V(C) \subseteq e$, then $C \preceq G$, and since $G \in \text{Forb}(\mathcal{A})$, it follows that $C \notin \mathcal{A}$. Otherwise, there are two vertices, say v_1 and v_2, in $V(C)$ not incident with the same edge of H. Since C is 2-connected, these two vertices must be both on some cycle C' in C. The length of C' is at least the length of the shortest cycle in H, that is,

$$|V(C')| \geq \text{girth}(H) > q = \max\{|V(A)| : A \in \mathcal{A}\}.$$

Therefore, $|V(C)| > \max\{|V(A)| : A \in \mathcal{A}\}$, and thus $C \notin \mathcal{A}$.

Therefore it suffices to show that $F \overset{\text{ind}}{\longrightarrow} (G)_r^{K_1}$. Let $\Delta : V(F) \to [1,r]$. Since $V(F) = V(H)$, and $\chi(H) > r$, there exists $e \in E(H)$ such that e is monochromatic under Δ. However, $F[e]$ is then a monochromatic copy of G. □

8.4 Triangle-free graphs

The following theorem is a traditional example and a nice "nugget" from the field of Restricted Ramsey theory.

Theorem 8.4.1 (Nešetřil and Rödl, 1975 [104]). *Let $r \in \mathbb{Z}^+$. For every $G \in$ Forb(K_3), there exists an $F \in$ Forb(K_3) such that*

$$F \overset{ind}{\longrightarrow} (G)_r^{K_2}.$$

In fact, much more can be proved. Recall that a *clique* in a graph is a subgraph which is complete, and the *clique number* of a graph G, denoted $cl(G)$, is the size of the largest clique in G.

Theorem 8.4.2 (Nešetřil and Rödl, 1981 [109]). *Let $r, a \in \mathbb{Z}^+$. For every $G \in$ Forb(K_a), there exists a graph $F \in$ Forb(K_a) such that*

$$F \overset{ind}{\longrightarrow} (G)_r^{K_2}.$$

Outline of proof. Follow the proof of Theorem 5.5.4. Note that $cl(P^0) = cl(G)$, and observe that the inductive construction and the amalgamations taking place at each step preserve the clique number. Therefore, for every $i \in [1, \binom{s}{2}]$, $cl(P^i) = cl(P^{i-1})$, and since $P^{\binom{s}{2}} \xrightarrow{\text{ind}} (G)_r^{K_2}$, $P^{\binom{s}{2}}$ is the desired graph. □

The following theorem exploits the proof of Theorem 5.5.4 even more:

Theorem 8.4.3 (Nešetřil and Rödl, 1981 [109])**.** *For any $r, m \in \mathbb{Z}^+$, there exists a graph F such that $F \longrightarrow (K_m)_r^{K_2}$, and any two copies of K_m in F intersect in at most one edge.*

Proof. Again apply the construction in the proof of Theorem 5.5.4 with $G = K_m$, and notice that the inductive construction preserves the property that any two complete graphs intersect in at most two vertices. □

Chapter 9

Ramsey minimal graphs

9.1 Definitions

In this section, all colourings are edge colourings in two colours. For graphs G_1, G_2, a graph F is *weakly* (G_1, G_2)*-minimal* iff both $F \longrightarrow (G_1, G_2)_2^{K_2}$, and for any proper subgraph $F' \subsetneq F$, $F' \not\longrightarrow (G_1, G_2)_2^{K_2}$. Let $\mathcal{R}_{\min}(G_1, G_2)$ denote the family of all weakly (G_1, G_2)-minimal graphs. The graph F is *strongly* (G_1, G_2)*-minimal* iff both $F \xrightarrow{\text{ind}} (G_1, G_2)_2^{K_2}$, and for any proper subgraph $F' \subsetneq F$, $F' \overset{\text{ind}}{\not\longrightarrow} (G_1, G_2)_2^{K_2}$. Let $\mathcal{R}^*_{\min}(G_1, G_2)$ denote the family of all strongly (G_1, G_2)-minimal graphs.

A pair of graphs (G_1, G_2) is *weakly (resp.* strongly*) Ramsey-finite* iff $\mathcal{R}_{\min}(G_1, G_2)$

(resp. $\mathcal{R}^*_{\min}(G_1, G_2)$) is finite and *weakly (resp.* strongly*) Ramsey-infinite* otherwise. Consider the following example:

Theorem 9.1.1 (Burr, Erdős, Faudree, Rousseau, and Schelp, 1981 [18])**.** *For all $k \in \mathbb{Z}^+$, C_{2k+1} is both strongly and weakly (P_2, P_2)-minimal, and therefore (P_2, P_2) is both strongly and weakly Ramsey-infinite.*

Proof. In both the weak and strong cases the same proof works, so only the proof of the weak case is presented here. Let $k \in \mathbb{Z}^+$, and let $\Delta : E(C_{2k+1}) \to \{\text{red}, \text{blue}\}$. Then there must be two adjacent edges that receive the same colour. Therefore $C_{2k+1} \longrightarrow (P_2, P_2)_2^{K_2}$.

If any edge is removed from C_{2k+1}, the remaining edges can be alternately coloured red and blue, producing no monochromatic P_2, and since $\delta(C_{2k+1}) > 0$, removing any vertex removes an edge as well, proving that C_{2k+1} is weakly (P_2, P_2)-minimal. $\square$

Characterizing the pairs of graphs that are Ramsey-finite (or Ramsey-infinite) was a problem asked by Nešetřil and Rödl in the mid 1970's (see [37]). Today this problem is central to the subfield of Ramsey minimal graphs.

9.2 Classic minimal graph Ramsey results

One of the earliest theorems in the field of Ramsey minimal graphs was motivated by a conjecture due to Nešetřil, namely that for any $a, b \in \mathbb{Z}^+$, the pair (K_a, K_b) is Ramsey-infinite (both strongly and weakly, since in this case all results are both weak and strong).

Theorem 9.2.1 (Burr, Erdős and Lovász, 1976 [21]). *For all $a, b \geq 3$, and $d \in \mathbb{Z}^+$, there exists $F \in \mathcal{R}_{min}(K_a, K_b)$ such that $\Delta(F) \geq d$. Therefore (K_a, K_b) is Ramsey-infinite (both weakly and strongly).*

Some widely cited results in the field of Ramsey minimal graphs are the following theorems due to Nešetřil and Rödl (published in 1978). Recall that for any $n \in \mathbb{Z}^+$, a graph is said to be n-*connected* if at least n points must be removed to disconnect the graph (note that every connected graph is at least 1-connected). A graph is said to be 2.5-*connected* iff it is 2-connected, and the removing any two points connected by an edge does not disconnect the graph.

Theorem 9.2.2 (Nešetřil and Rödl, 1978 [107]). *For every 2.5-connected graph G, (G, G) is strongly Ramsey-infinite.*

Theorem 9.2.3 (Nešetřil and Rödl, 1978 [107]). *For every 3-connected graph G, (G, G) is weakly Ramsey-infinite.*

Theorem 9.2.4 (Nešetřil and Rödl, 1978 [107]). *For every graph G with $\chi(G) \geq 3$, (G, G) is both strongly and weakly Ramsey-infinite.*

9.3 Weakly Ramsey minimal results for matchings

In the same paper as Theorem 9.2.1, Burr *et al.* provide some of the first Ramsey finite pairs.

Theorem 9.3.1 (Burr, Erdős and Lovász, 1976 [21]). *$\mathcal{R}_{min}(2K_2, 2K_2) = \{3K_2, C_5\}$, and for all $m, n \in \mathbb{Z}^+$, (mK_2, nK_2) is weakly Ramsey-finite.*

Burr *et al.* prove Theorem 9.3.1 by showing essentially that any minimal graph that arrows $(mK_2, nK_2)_2^{K_2}$ has at most $2(m+n)^2$ vertices, proving that there are only finitely many such graphs.

Two years later, Theorem 9.3.1 was generalized to only require one of the two graphs be a matching.

Theorem 9.3.2 (Burr, Erdős, Faudree, and Schelp, 1978 [20]). *For any $m \in \mathbb{Z}^+$, and any finite graph G, (mK_2, G) is weakly Ramsey-finite.*

Theorem 9.3.2 is proven using a similar argument as the proof of Theorem 9.3.1: it is shown that there exists $c = c(m, G)$ such that any minimal graph F such that $F \longrightarrow (mK_2, G)_r^{K_2}$ has at most c edges which, together with the fact that any such minimal graph has no isolated vertices, proves that there are only finitely many. The authors also suggest the following conjecture which is still open today:

Conjecture 9.3.3 (Burr, Erdős, Faudree, and Schelp 1978 [20]). *Let G be a graph. If for all graphs H, (G, H) is weakly Ramsey-finite, then there exists $m \in \mathbb{Z}^+$ such that $G \cong mK_2$.*

The following theorems show progress towards Conjecture 9.3.3, and follow from two corollaries due to Łuczak in 1994 [92]:

Theorem 9.3.4. *Let G be a graph. If for any forest H that is not a disjoint union of stars, (G, H) is weakly Ramsey-finite, then there exists $m \in \mathbb{Z}^+$ such that $G \cong mK_2$.*

Theorem 9.3.5. *Let G be a graph. If for any $m \in \mathbb{Z}^+$, $(G, K_{1,2m})$ is weakly Ramsey-finite, then there exists $m \in \mathbb{Z}^+$ such that $G \cong mK_2$.*

9.4 Weakly Ramsey minimal results for stars and forests

By Theorem 9.2.3, if G is 3-connected, $\mathcal{R}_{\min}(G, G)$ is infinite, and by Theorem 9.2.4, if G contains an odd cycle, $\mathcal{R}_{\min}(G, G)$ is again infinite. When one of both of G_1 and

G_2 is acyclic, $\mathcal{R}_{\min}(G_1, G_2)$ may be finite or infinite. So, *e.g.*, in the next theorem, when $G_1 = G_2$ are both odd stars, then $\mathcal{R}_{\min}(G, G)$ is finite.

Theorem 9.4.1 (Burr, Erdős and Lovász, 1976 [21]). *For any graph F,*

$$F \longrightarrow (K_{1,n}, K_{1,n})_2^{K_2}$$

iff $\Delta(F) \geq 2n-1$ or, if n is even, F contains a component which is regular of degree $2n-2$ with an odd number of points.

Note that Theorem 9.4.1 implies (since $F = K_{1,2n-1}$ is the smallest graph such that $\Delta(F) \geq 2n-1$) that for odd n, $\mathcal{R}_{\min}(K_{1,n}, K_{1,n}) = \{K_{1,2n-1}\}$. For n even, Theorem 9.4.1 does not immediately reveal if $\mathcal{R}_{\min}(K_{1,n}, K_{1,n})$ is finite or infinite.

Theorem 9.4.2 (Burr, Erdős, Faudree, Rousseau, Schelp, 1981 [18]). *Let $s, t \in \mathbb{Z}^+$. If both s and t are odd, $\mathcal{R}_{min}(K_{1,s}, K_{1,t}) = \{K_{1,s+t-1}\}$. Otherwise, $\mathcal{R}_{min}(K_{1,s}, K_{1,t})$ is infinite.*

In fact, Burr *et al.* proved more than Theorem 9.4.2. Call a star $K_{1,n}$ *non-trivial* if $n > 1$.

Theorem 9.4.3 (Burr, Erdős, Faudree, Rousseau, Schelp, 1981 [18]). *If G_1 and G_2 are unions of non-trivial stars, then $\mathcal{R}_{min}(G_1, G_2)$ is finite if and only if G_1 and G_2 are both odd stars.*

In 1991, Faudree classified for which forests G_1 and G_2 the pair (G_1, G_2) is weakly Ramsey-finite.

Theorem 9.4.4 (Faudree, 1991 [51]). *For any two forests G_1 and G_2, (G_1, G_2) is weakly Ramsey-finite if and only if one of the following holds:*

1. *One of G_1 and G_2 is a matching (see Theorem 9.3.2),*

2. *There exist $x, y \in \mathbb{Z}^+$, and odd integers m, n such that $G_1 \cong K_{1,m} \cup xK_2$ and $G_2 \cong K_{1,n} \cup yK_2$,*

3. *There exist $x, y \in \mathbb{Z}^+$, x sufficiently large, $s, n, m_1, \ldots, m_s \in \mathbb{Z}^+$, with n and m_1 odd, and $m_1 \geq n + m_2 - 1$, such that $G_1 \cong K_{1,n} \cup xK_2$, and $G_2 \cong \bigcup_{i=1}^{s} K_{1,m_i} \cup yK_2$*

Theorem 9.4.5 (Burr, Erdős, Faudree, Rousseau, Schelp, 1980 [17]). *Let $n, k \in \mathbb{Z}^+$, and let $\{G_1, \ldots, G_n\}$ be any set of 2-connected graphs. Then $\mathcal{R}_{min}(\bigcup_{i=1}^{n} G_i, K_{1,k})$ is infinite.*

Łuczak later showed that as soon as one of the graphs is no longer a forest, the pair is weakly Ramsey-infinite.

Theorem 9.4.6 (Łuczak, 1994 [92]). *For any graph G_1 containing at least one cycle, and for any forest G_2 which is not a matching, the set $\mathcal{R}_{min}(G_1, G_2)$ is infinite.*

9.5 Explicit sets of Ramsey minimal graphs

After determining whether the pair (G_1, G_2) is (weakly or strongly) Ramsey-infinite or not, one can ask to classify the elements of the sets $\mathcal{R}_{\min}(G_1, G_2)$ and $\mathcal{R}^*_{\min}(G_1, G_2)$. Only very few results of this type are known. Consider the following example:

Theorem 9.5.1 (see, *e.g.*, [20]). *$\mathcal{R}_{min}(K_2, H) = \mathcal{R}^*_{min}(K_2, H) = \{H\}$.*

Proof. Firstly, note that $H \in \mathcal{R}_{\min}(K_2, H) \cap \mathcal{R}^*_{\min}(K_2, H)$. Secondly, note that any graph that arrows $(K_2, H)_2^{K_2}$ (either weak or induced) must contain a copy (weak or induced) of H. □

Burr *et al.* [21] proved that $\mathcal{R}_{\min}(2K_2, 2K_2) = \{3K_2, C_5\}$ (see Theorem 9.3.1), and later [18] that when s and t are odd, $\mathcal{R}_{\min}(K_{1,s}, K_{1,t}) = \{K_{1,s+t-1}\}$ (see Theorem 9.4.2).

Theorem 9.5.2 (Burr, Erdős, Faudree, and Schelp, 1978 [20]). *The graphs in Figure 9.1 are the only weakly Ramsey-minimal graphs for $(2K_2, K_3)$.*

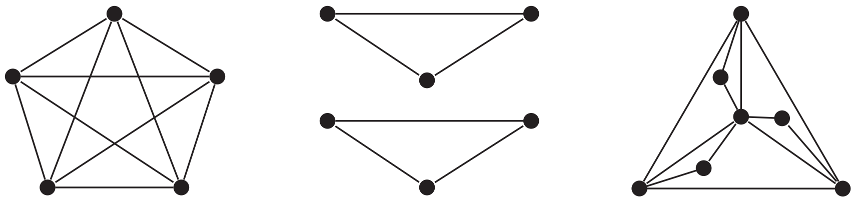


Figure 9.1: The members of $\mathcal{R}_{\min}(2K_2, K_3)$.

In the last ten years, the following weakly Ramsey-minimal sets have been characterized:

(a) $\mathcal{R}_{\min}(2K_2, K_{1,n})$ (see [99]),

(b) $\mathcal{R}_{\min}(2K_2, tK_2)$ (see [100]),

(c) $\mathcal{R}_{\min}(K_{1,2}, K_{1,m})$ (see [11]),

(d) $\mathcal{R}_{\min}(K_{1,2}, K_3)$ (see [12]).

In [99], the following sets are also explicitly determined:

$$\mathcal{R}_{\min}(2K_2, K_{1,2}) = \{2K_{1,2}, C_4, C_5\},$$

and $\mathcal{R}_{\min}(2K_2, K_{1,3})$ is exactly the collection of graphs in Figure 9.2.

In [100], for $t \leq 5$, the sets $\mathcal{R}_{\min}(2K_2, tK_2)$ are also explicitly determined. Borowiecki *et al.* [11], in addition to classifying the elements of $\mathcal{R}_{\min}(K_{1,2}, K_{1,m})$, produce, for m and n odd integers, a sufficient condition for a graph to belong to $\mathcal{R}_{\min}(K_{1,m}, K_{1,n})$.

9.6 Minimal ordered Ramsey graphs

Let $(F, \leq)$ and $(G, \leq)$ be ordered graphs, and assume that $(F, \leq) \xrightarrow{\text{ind}} (G, \leq)_2^{K_2}$. Then F is said to be *minimal ordered Ramsey* iff for every proper subgraph $F' \subsetneq F$,

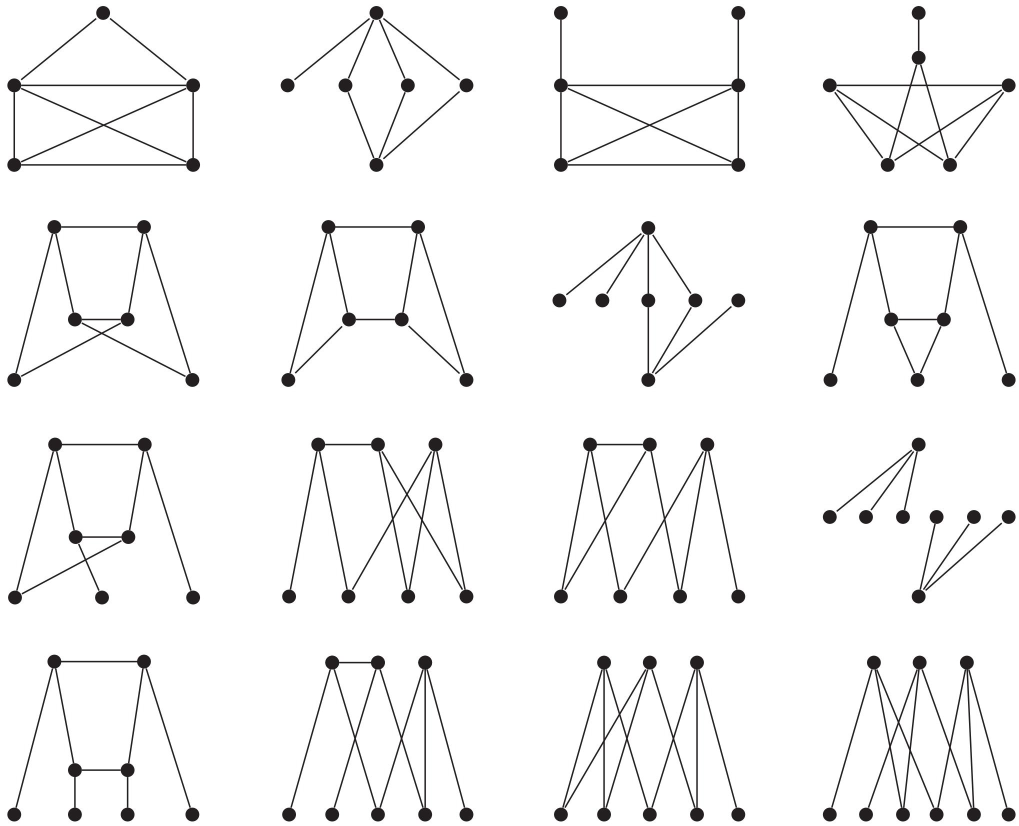


Figure 9.2: The members of $\mathcal{R}_{\min}(2K_2, K_{1,3})$.

$(F', \leq) \overset{\text{ind}}{\not\longrightarrow} (G, \leq)_2^{K_2}$. In 1990, Gunderson [68] proposed the problem of finding minimal ordered Ramsey graphs, and presented a few trivial examples, together with the following first non-trivial example:

Theorem 9.6.1 (Gunderson, 1990 [68]). *Let $(G, \leq)$ and $(F, \leq)$ be as in Figure 9.3. Then $(F, \leq)$ is minimal ordered Ramsey for $(G, \leq)$.*

For any two graphs G_1, G_2, let $\hat{r}_{\text{ind}}(G_1, G_2)$ denote the least integer e such that there exists a graph F with exactly e edges such that $F \overset{\text{ind}}{\longrightarrow} (G_1, G_2)_2^{K_2}$. The number

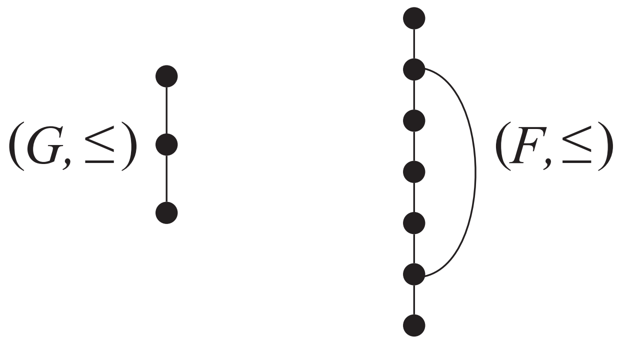


Figure 9.3: A non-trivial minimal ordered Ramsey graph $(F, \leq)$ for $(G, \leq)$

$\hat{r}_{\text{ind}}(G_1, G_2)$ is the *size Ramsey number*. The following conjecture arose from the study of minimal ordered Ramsey graphs.

Conjecture 9.6.2 (Gunderson [68]). *Let F and G be finite graphs, and assume that $F \stackrel{ind}{\longrightarrow} (G)_2^{K_2}$. Let G' be such that $G \subsetneq G'$ and $V(G) = V(G')$. Then if either $|V(F)| = R_{ind}(G, G)$, or $|E(F)| = \hat{r}_{ind}(G, G)$, then $G' \not\leq F$.*

9.7 Minimal Ramsey minimum degree

Recently, Fox and Lin [54] considered the following Ramsey-minimal problem: for any finite graph G, what is the value of $s(G) = \min\{\delta(F) : F \in \mathcal{R}_{\min}(G, G)\}$? They produced the following theorems:

Theorem 9.7.1 (Fox and Lin, 2007 [54]). *For all graphs G,*

$$2\delta(G) - 1 \leq s(G) \leq R(G, G) - 1.$$

Theorem 9.7.2 (Fox and Lin, 2007 [54]). *For all* $k \in \mathbb{Z}^+$*,* $s(K_k) = (k-1)^2$*, and for all* $a, b \in \mathbb{Z}^+$*,* $s(K_{a,b}) = 2\min\{a, b\} - 1$*.*

Chapter 10

Colouring incomplete graphs

10.1 Introduction

So far in this thesis it has been shown (Observation 4.1.1 and Theorem 5.7.3 respectively) that for any $r \in \mathbb{Z}^+$, and any graphs G and H, the numbers $R(G; H; r)$ and $R_{\text{ind}}(G; H; r)$ exist when H is complete; that is, when H is complete, for any graph G, there exists graphs F_1 and F_2 such that $F_1 \longrightarrow (G)_r^H$, and $F_2 \xrightarrow{\text{ind}} (G)_r^H$. When H is not complete, the situation is more complicated, as exhibited by the following theorem.

Theorem 10.1.1. *For any graph F, $F \not\rightarrow (C_4)_2^{P_2}$, and $F \overset{ind}{\not\longrightarrow} (C_4)_2^{P_2}$. Thus $R(C_4, P_2; 2)$ and $R_{ind}(C_4; P_2; 2)$ do not exist.*

Proof. The same proof works for both the induced and the non-induced cases, so only the non-induced case is presented here. Let F be any graph. Define a total (linear) ordering " $<$ " on $V(F)$ in any way. Define a 2-colouring $\Delta : \binom{F}{P_2} \to \{\text{red}, \text{blue}\}$ as follows: let $P_2' \in \binom{F}{P_2}$, and let $x_1, \{x_1, x_2\}, x_2, \{x_2, x_3\}, x_3$ be the vertices and edges of P_2'. If $x_2 > x_1$ and $x_2 > x_3$ (in the fixed but arbitrary ordering) then let $\Delta(P_2') = \text{red}$. If $x_2 < x_1$ and $x_2 < x_3$ then let $\Delta(P_2') = \text{blue}$. Otherwise, the colour does not matter (let $\Delta(P_2') = \text{red}$, say).

Let $C_4' \in \binom{F}{C_4}$, and assume $V(C_4') = \{v_1, v_2, v_3, v_4\}$, and v_1 and v_4 are the smallest and greatest elements respectively with respect to the total linear ordering on F. By the definition of Δ, the copy of P_2 inside C_4' with v_1 as its middle point must be blue, and the copy of P_2 with v_4 in the middle must be coloured red. Therefore C_4' contains both a red copy of P_2 and a blue copy of P_2, and since C_4' was an arbitrary copy of C_4 in F, every C_4 in F contains two colours. Finally, since the colouring was defined for any arbitrary ordering, and one can always order any finite graph, the result follows. □

This problem can in fact be found whenever H is non-empty and non-complete.

Theorem 10.1.2 (Nešetřil and Rödl, 1975 [103])**.** *Let H be a non-empty, non-complete graph. Then there exists a graph G such that $R(G; H; 2)$ does not exist.*

Proof. (as given in [121, p. 195]) Note that since H is neither complete nor empty, there are at least two orderings $\leq_1$ and $\leq_2$ of $V(H)$ such that $(H, \leq_1)$ and $(H, \leq_2)$ are non-isomorphic. By Theorem 5.8.3, let G be a finite graph such that $G \xrightarrow{\text{ind}}_{\text{ord}} (H, \leq_1)$, and $G \xrightarrow{\text{ind}}_{\text{ord}} (H, \leq_2)$.

Let F be any graph, and let $\leq$ be an arbitrary ordering of $V(F)$. Define $\Delta : \binom{F}{H} \to \{\text{red}, \text{blue}\}$ by

$$\Delta(H') = \begin{cases} \text{red if } (H', \leq) \cong (H', \leq_1), \text{ and} \\ \\ \text{blue otherwise.} \end{cases}$$

Then, by the definition of G, each $G' \in \binom{F}{G}$ contains copies of $(H, \leq_1)$ and $(H, \leq_2)$, and therefore is not monochromatic under Δ. $\square$

10.2 A classification

The problem of classifying which triples (G, H, r) can F be found such that $F \xrightarrow{\text{ind}} (G)^H_r$ was addressed by Gunderson, Rödl and Sauer in 1990 [69]. The notation in this section is due to Gunderson. Given a hypergraph H, define

$$ORD(H) = \{(H, \leq_1), (H, \leq_2), \ldots, (H, \leq_k)\}$$

to be the set of all distinct ordered hypergraphs which, without the ordering, are copies of H, that is, H together with every possible ordering of $V(H)$. For example, $ORD(P_2)$ is the set of all possible orderings of P_2, given in Figure 10.1, and $ORD(C_4)$ is given in Figure 10.2.

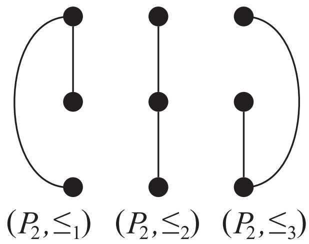


Figure 10.1: The non-isomorphic orderings of P_2

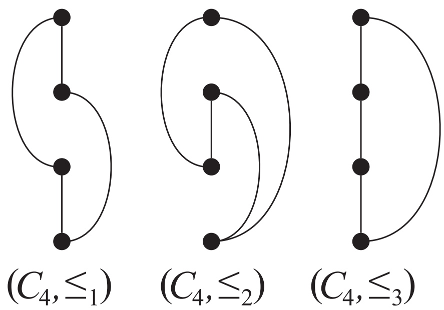


Figure 10.2: The non-isomorphic orderings of C_4

Given H and $(G, \leq^*)$, let

$$DO(H, G, \leq^*) = \{(H, \leq) \in ORD(H) : \binom{G, \leq^*}{H, \leq} \neq \emptyset\}$$

be the set of distinct orderings on H that can be found as induced subgraphs in $(G, \leq^*)$. For example,

$$DO(P_2, C_4, \leq_2) = \{(P_2, \leq_1), (P_2, \leq_3)\}.$$

Finally, let

$$\mathrm{mdo}(H,G) = \min\{|DO(H,G,\leq_j)| : (G,\leq_j) \in ORD(G)\}$$

be the minimum number of distinct orderings on H given any ordering of G. For example, $\mathrm{mdo}(P_2, C_4) = 2$.

For another example, given some $n \in \mathbb{Z}^+$, and some graph G, consider the set $DO(K_n, G, \leq_j)$. If G does not contain a copy of K_n, then $DO(K_n, G, \leq_j) = \emptyset$. If G does contain a copy of K_n, then since all orderings on complete graphs are isomorphic, $|ORD(K_n)| = 1$, and so $|DO(K_n, G, \leq_j)| = 1$ and $\mathrm{mdo}(K_n, G) \leq 1$.

The reason that there does not exist a graph F such that $F \xrightarrow{\text{ind}} (C_4)_2^{P_2}$ is because $\mathrm{mdo}(P_2, C_4) = 2$, which means there is a way to colour the P_2's according to their ordering (or order type) to ensure that every C_4 contains two colours. The following theorem demonstrates this fact, and is proved directly using the Ordered Hypergraph theorem (Theorem 5.8.1).

Theorem 10.2.1 (Prömel and Voigt, 1985 [120]). *Given $r \in \mathbb{Z}^+$, and G, H graphs such that $mdo(H,G) = 1$, then there exists a graph F such that*

$$F \xrightarrow{ind} (G)_r^H.$$

Proof. Let $r \in \mathbb{Z}^+$, and let G and H be graphs such that $\mathrm{mdo}(H,G) = 1$. By the definition of $\mathrm{mdo}(H,G)$, there exists an ordering on G, say $\leq$, so that

$|DO(H, G, \leq)| = 1$ (every induced H-subgraph is order-isomorphic to, say, $(H, \leq)$). By the Ordered Hypergraph theorem, there exists an ordered graph $(F, \leq)$ such that $(F, \leq) \xrightarrow{\text{ind}} (G, \leq)_r^{(H,\leq)}$. The claim is then that F, the unordered version of $(F, \leq)$, satisfies

$$F \xrightarrow{\text{ind}} (G)_r^H.$$

Let $\Delta : \binom{F}{H}_{\text{ind}} \to \{1, \ldots, r\}$. Define an r-colouring $\Delta^* : \binom{F,\leq}{H,\leq}_{\text{ind}} \to \{1, \ldots, r\}$ defined by $\Delta^*((H', \leq)) = \Delta(H')$. By the choice of F, there exists a monochromatic $(G^*, \leq) \in \binom{F,\leq}{G,\leq}_{\text{ind}}$ such that Δ^* is constant on $\binom{G^*,\leq}{H,\leq}_{\text{ind}}$. Since $|DO(H, G, \leq)| = 1$, $(H', \leq) \in \binom{G^*,\leq}{H,\leq}_{\text{ind}}$ iff $H' \in \binom{G^*}{H}$, and therefore Δ is constant on $\binom{G^*}{H}_{\text{ind}}$. □

However, Theorem 10.2.1 appears to be far from an if and only if statement, as shown by the following theorem:

Theorem 10.2.2 (Gunderson, Rödl, and Sauer, 1990 [69]). *For every non-complete, non-empty graph H, if either H or $\overline{H}$ is 2-connected, then there exist graphs F and G such that $mdo(H, G) \geq 2$, and $F \xrightarrow{\text{ind}} (G)_r^H$.*

Gunderson, Rödl and Sauer succeeded in characterizing all triples (G, H, r) such that there exists a graph F with $F \xrightarrow{\text{ind}} (G)_r^H$. Given two hypergraphs G, H, define the hypergraph $S_{H,G}$ as the graph with vertex set $ORD(H)$ and edge set $E(S_{H,G}) = \{DO(H, G, \leq) : (G, \leq) \in ORD(G)\}$. Let $\chi(S_{H,G})$ denote the (weak) chromatic number of $S_{H,G}$.

Theorem 10.2.3 (Gunderson, Rödl, and Sauer, 1990 [69]). *For any hypergraphs G, H, there exists F such that $F \xrightarrow{ind} (G)^H_r$ iff $\chi(S_{H,G}) > r$.*

Three years later, Gunderson, Rödl, and Sauer published [70] a theorem characterizing explicitly the set of graphs G for which there exists a graph F such that $F \xrightarrow{\text{ind}} (G)^{P_2}_2$.

Appendix A

Graph Theory

Throughout this thesis, unless otherwise noted, all graphs are finite. A graph G is an ordered pair $(V(G), E(G))$, where $V = V(G)$ is a set, and $E = E(G) \subseteq [V]^2$. Let G, H be graphs. An *isomorphism between* G *and* H is a bijection $f : V(G) \to V(H)$ such that $\{x, y\} \in E(G)$ iff $\{f(x), f(y)\} \in E(H)$. If there exists an isomorphism between two graphs G and H, say G and H are *isomorphic*, and write $G \cong H$. A graph H is said to be a *subgraph* of G, denoted $H \subseteq G$, iff $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G) \cap [V(H)]^2$. For any two graphs G and H, an *embedding of* H *into* $G$is a function $f : V(H) \to V(G)$ such that for each $\{v_1, v_2\} \in E(H)$, $\{f(v_1), f(v_2)\} \in E(G)$, and a *copy of* H *in* G is an image of such an embedding.

The set of all copies of H in G is denoted

$$\binom{G}{H}.$$

The graph H is said to be an *induced subgraph* of G, denoted $H \preceq G$, iff $V(H) \subseteq V(G)$ and $E(H) = E(G) \cap [V(H)]^2$. For any two graphs G and H, an *induced embedding of H into G* is a function $f : V(H) \to V(G)$ such that $\{v_1, v_2\} \in E(H)$ iff $\{f(v_1), f(v_2)\} \in E(G)$, and an *induced copy of H in G* is an image of such an induced embedding. The set of all induced copies of H in G is denoted

$$\binom{G}{H}_{\text{ind}}.$$

A graph G is *complete* if every pair of vertices in G is connected by an edge; the complete graph on n vertices is denoted by K_n. A graph G is *k-partite* if there exists a partition $V(G) = V_1 \cup \cdots \cup V_k$ such that for every $\{x, y\} \in E(G)$, there exist $i, j \in [1, k]$, $i \neq j$, such that $x \in V_i$ and $y \in V_j$. A 2-partite graph is called *bipartite*. For any $m, n \in \mathbb{Z}^+$, the *complete bipartite graph $K_{m,n}$* is the bipartite graph with partite sets A and B containing m and n elements respectively, and $E(K_{m,n}) = \{\{a, b\} : a \in A, b \in B\}$.

For any graph G, and any vertex $v \in V(G)$, the *degree of v in G* is

$$\deg_G(v) = |\{e \in E(G) : v \in e\}|.$$

The *neighbourhood of v in G*, denoted $N_G(v)$, is the set $N_G(v) = \{x \in V(G) : \{x, v\} \in E(G)\}$. Note that for all $v \in V(G)$, $\deg_G(v) = |N_G(v)|$. When it is clear which graph the degree or the neighbourhood is being considered in, the "G" may be omitted. The following lemma is a traditional lemma in Graph theory (see *e.g.* [7]).

Lemma A.0.4 (Handshaking Lemma)**.** *For any graph G,*

$$\sum_{v \in V(G)} \deg(v) = 2|E(G)|.$$

For any $n \in \mathbb{Z}^+$, a *path of length n* in a graph G is an alternating sequence of distinct vertices and edges $v_1, e_1, v_2, e_2, \ldots, e_n, v_{n+1}$ such that for all $i \in [1, n]$, $e_i = \{v_i, v_{i+1}\}$. For two vertices u and v, a*uv-path* is a path starting at u and ending at v. A *cycle of length n* in G is an alternating sequence of distinct vertices and edges of the form $v_1, e_1, \ldots, v_n, e_n, v_1$, such that for all $i \in [1, n-1]$, $e_i = \{v_i, v_{i+1}\}$, and $e_n = \{v_n, v_1\}$. For any $n \in \mathbb{Z}^+$, P_n denotes a path on n edges (note that there is some disagreement in graph theory as to whether P_n should denote a path with n edges or a path with n vertices). Let C_n denotes a cycle on n edges. For any graph G, the *complement of G* is the graph $\overline{G} = (V(G), [V(G)]^2 \setminus E(G))$.

A graph G is *connected* iff for any $u, v \in V(G)$, there is a uv-path in G. A *component* of G is a maximal connected subgraph of G.

A *forest* is a acyclic graph, the components of which are called *trees*. A *leaf* in a tree is a vertex of degree one. A tree is said to be *locally finite* if the degree of every vertex is finite. A *rooted tree* is a tree with some vertex identified as the *root*, and a *branch* in a rooted tree T with root x is a component in the graph formed by removing x from T.

A set X of vertices in a graph G is said to be an *independent set* iff X does not contain any edge, *i.e.*,

$$E(G) \cap [X]^2 = \emptyset.$$

The *independence number of* G, denoted $\alpha(G)$, is the cardinality of the largest independent set of G.

A *clique* in a graph is a complete subgraph. For $r \in \mathbb{Z}^+$, a *good vertex* r*-colouring* of a graph G is an r-colouring of $V(G)$ with the added property that no edge is completely contained within a colour class. *i.e.*, a good r-colouring is a partition of $V(G) = V_1 \cup \ldots \cup V_r$, with the added property that for all $\{x, y\} \in E(G)$, there exist i, j, $i \neq j$, such that $x \in V_i$ and $y \in V_j$. If for some graph G, there exists a good vertex r-colouring, then G is said to be r*-colourable*. When it causes no confusion, a good vertex r-colouring is also referred to as simply a good r-colouring. The *chromatic number* of G, denoted $\chi(G)$, is the smallest r such that G is r-colourable.

For graphs G and H such that $V(G) \cap V(H) = \emptyset$, the *disjoint union of* G *and* H, denoted $G \dot{\cup} H$, is the graph $(V(G) \cup V(H), E(G) \cup E(H))$. This definition can be extended to graphs who have vertices in common as follows: for any graphs G and

H, $V(G \dot\cup H) = (V(G) \times \{0\}) \cup (V(H) \times \{1\})$, and

$$\begin{aligned} E(G \dot\cup H) &= \{ \ \{(x,0),(y,0)\} : \{x,y\} \in E(G) \ \} \cup \\ & \quad \{ \ \{(x,1),(y,1)\} : \{x,y\} \in E(H) \ \}. \end{aligned}$$

For any two graphs G and H, let $G+H$ denote the graph with $V(G+H) = V(G \dot\cup H)$, and

$$E(G+H) = E(G \dot\cup H) \cup \{ \ \{(a,0),(b,1)\} : a \in V(G), b \in V(H)\}$$

(that is, the disjoint union of G and H, together with every vertex in G connected to every vertex in H).

For any connected graph G, and any $n \in \mathbb{Z}^+$, let nG denote the graph with n components, each isomorphic to G. So, *e.g.,* mK_2 is a collection of m disjoint edges–called a *matching*.

For any graph G and $v \in V(G)$, let $G \setminus \{v\}$ denote the graph formed by removing v, and all edges attached to v, from G. If for some graph G, there exists a graph G' such that for any edge $e \in E(G)$, $G' \cong (V(G), E(G) \setminus \{e\})$, then denote G' by $G \setminus e$. Define $G + e$ analogously. Similarly, if for some graphs G and H, there exists a graph G' such that for any $H' \in \binom{G}{H}$, $G' \cong (V(G), E(G) \setminus E(H'))$, then denote G' by $G \setminus E(H)$. For example, $K_5 \setminus P_2$ is the graph formed by removing two incident edges from K_5.

A *hypergraph* G is an ordered pair of sets $(V(G), E(G))$, where $E = E(G) \subseteq \mathcal{P}(V)$, the collection of all subsets of $V = V(G)$. As with graphs, the elements of $V = V(G)$ are called *vertices*, but the elements of $E = E(G)$ are called *hyperedges*. Note that by this definition, singletons may be edges. To avoid trivialities, singletons are usually not allowed to be edges in hypergraphs. Note that when helpful a script notation can be used to denote a hypergraph (*e.g.*, $\mathcal{G} or \mathcal{H}$) to differentiate graphs from hypergraphs. For $k \in \mathbb{Z}^+$, a hypergraph H is said to be *k-uniform* if $E \subseteq [V]^k$. By definition, a graph is a 2-uniform hypergraph. Isomorphisms between hypergraphs are defined analogously to those on graphs, as are subhypergraphs (induced and non-induced).

For any hypergraph G, and any vertex $v \in V(G)$, the *degree of v in G*, written $\deg_G(v)$, is $\deg_G(v) = |\{e \in E(G) : v \in e\}|$. The *neighbourhood of v in G*, denoted $N_G(v)$, is the set $N_G(v) = \{x \in V(G) : \exists e \in E(G) \text{ s.t. } \{x, v\} \subseteq e\}$. Note that when G is a hypergraph, it is not necessarily true that for all $v \in V(G)$, $\deg_G(v) = |N_G(v)|$ as is true in the graph case. For any $A \subseteq V(G)$, $A \neq \emptyset$, let $N_G(A) = \bigcap_{v \in A} N_G(v)$. When it is clear which hypergraph the degree or the neighbourhood is being considered in, the "G" may be omitted.

A hypergraph G is *complete* iff $E(G) = \mathcal{P}(G)$, and for any $k \in \mathbb{Z}^+$, a k-uniform hypergraph is *complete* if $E(G) = [V(G)]^k$.

A set of vertices X inside a hypergraph G is said to be an *independent set* iff $E(G) \cap \mathcal{P}(X) = \emptyset$. The *independence number of* G, denoted $\alpha(G)$, is the cardinality of the largest independent set of G.

Let $r \in \mathbb{Z}^+$. A *good vertex* r*-colouring* of a hypergraph G is an r-colouring of $V(G)$ with the added property that no edge is completely contained within a partition class. *i.e.*, a good r-colouring is a partition of $V(G) = V_1 \cup \cdots \cup V_r$, with the added property that for all $e \in E(G)$, there exist i, j, $i \neq j$, such that $e \cap V_i \neq \emptyset$ and $e \cap V_j \neq \emptyset$. If for some hypergraph G, there exists a good vertex r-colouring, then G is said to be r*-colourable*. When it causes no confusion, a good vertex r-colouring is also referred to as simply a good r-colouring. The *chromatic number* of G, denoted $\chi(G)$, is the smallest r such that G is r-colourable.

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